IV. Discrete-Time System Structures

In this chapter, we examine the structures of discrete-time system and how they affect the processing of signals. In particular, we first look at the characterisation of LTI systems and discuss specific cases such as linear phase, allpass, minimum phase and feedback systems. We then focus on systems called filters and examine several types of filters and how they can be implemented.

A. System Function

Given an LTI system with impulse response $h[n]$, the input and the output are related by the convolution sum given by

$$y[n] = h[n] * x[n]$$
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

This relationship can be described by

$$Y[e^{j\omega}] = H[e^{j\omega}] . X[e^{j\omega}],$$

where $H[e^{j\omega}]$ is the frequency response of the system computed with the discrete-time Fourier transform of $h[n]$. Alternatively, this relationship can be expressed in the z-domain as

$$Y[z] = H[z]. X[z]$$

Since an LTI system can be completely characterised by its impulse response $h[n]$, we will focus on its z-transform which is referred to as the system function:

$$H[z] = \sum_{n=-\infty}^{\infty} h[n]\, z^{-n}$$

The frequency response may be derived from the system function by evaluating $H[z]$ around the unit circle:

$$H[e^{j\omega}] = H[z] \Big|_{z = e^{j\omega}}$$

For LTI systems described by a linear constant coefficient difference equation given by

$$y[n] + \sum_{k=1}^{p} a[k]\, y[n-k] = \sum_{k=0}^{q} b[k] \cdot x[n-k],$$

the system function is a rational function of z:

$$H[z] = \frac{\sum_{k=0}^{q} b[k]\, z^{-k}}{1 + \sum_{k=1}^{p} a[k] . z^{-k}} = A \frac{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}{\prod_{k=1}^{p} (1 - \alpha_k . z^{-1})},$$

which is defined by the location of its poles α_k and zeros β_k. It should be observed that each term in the numerator

$$1 - \beta_k z^{-1} = \frac{z - \beta_k}{z}$$

contributes a zero to the system function at $z = \beta_k$ and a pole to the system function at $z = 0$. Similarly, each term in the denominator contributes a pole at $z = \alpha_k$ and a zero at $z = 0$. Therefore, including the poles and zeros that may lie at $z = 0$ or $z = \infty$, the number of zeros in $H[z]$ is equal to the number of poles.

If the impulse response $h[n]$ is real-valued, $H[z]$ is a conjugate symmetric function of z:

$$H[z] = H^*[z^*],$$

and the complex poles and zeros occur in conjugate symmetric pairs. For example, if there is a complex pole at $z = z_0$ there is also a complex pole at $z = z_0^*$.

i) Stability and Causality

Stability and causality impose some constraints on the system function of LTI systems.

The impulse response of a stable system must be absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

The frequency response of a type I linear phase filter can be expressed in the form

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{k=0}^{N/2} a[k] \cos(k\omega),$$

where $a[k] = 2h\left(\frac{N}{2} - k\right)$, $k = 1, 2, \ldots, \frac{N}{2}$ and $a[0] = h\left(\frac{N}{2}\right)$

Type II Linear Phase Filters:

A type II linear phase filter has a symmetric impulse response and N is odd. Therefore, the centre of symmetry of $h[n]$ occurs at the half-integer value $\alpha = N/2$, as illustrated by the fig below.

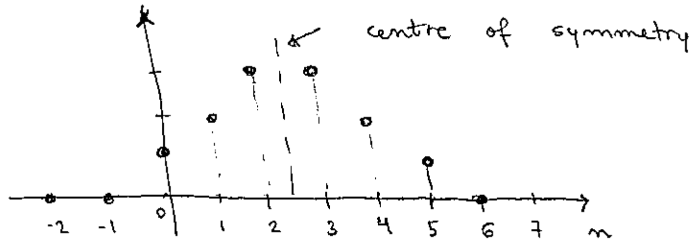


Fig. 4. Impulse response of a type II linear phase filter.

The frequency response of a type II linear phase filter can be written in the form

$$H(e^{j\omega}) = e^{-jN\omega/2} \sum_{k=1}^{(N+1)/2} b[k] \cos\left((k - 1/2)\omega\right),$$

where $b[k] = 2h\left(\frac{N+1}{2} - k\right)$, $k = 1, 2, \ldots, \frac{(N+1)}{2}$

Type III Linear Phase Filters:

A type III linear phase filter has an impulse response that is antisymmetric as given by:

$$h[n] = -h[N-n],$$

where N is even. Therefore, $h[n]$ is antisymmetric about $\alpha = N/2$ which is an integer as illustrated in the figure shown next.

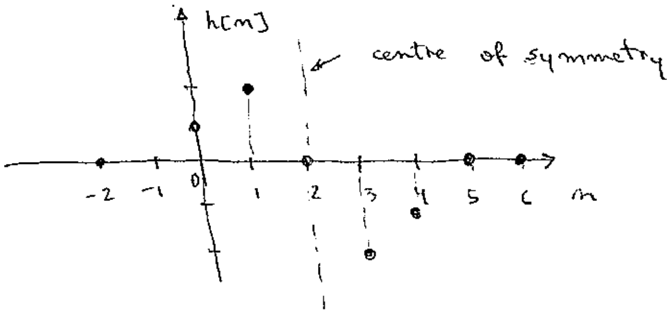


Fig. 5 Impulse response of a type III linear phase filter.

The frequency response of a type III linear phase filter can be written as

$$H(e^{j\omega}) = j\, e^{-jN\omega/2} \sum_{k=1}^{N/2} c[k] \cdot \sin(k\omega),$$

where $c[k] = 2\, h\left(\frac{N}{2} - k\right)$, $k = 1, 2, \ldots, \frac{N}{2}$.

Type IV Linear Phase Filters:

A type IV linear phase filter has an impulse response that is antisymmetric and N is odd. Therefore, $h[m]$ is antisymmetric about the half-integer value $\alpha = N/2$, as shown in the figure below.

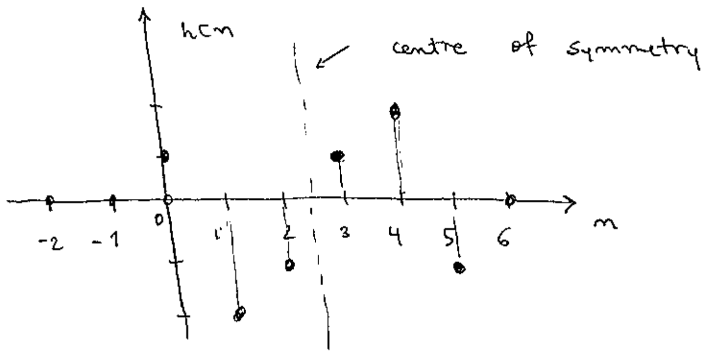


Fig. 6 Impulse response of a type IV linear phase filter.

The frequency response of a type IV linear phase filter has the form

$$H(e^{j\omega}) = j\, e^{-jN\omega/2} \sum_{k=1}^{(N+1)/2} d[k] \sin\left[(k - 1/2)\omega\right],$$

where $d[k] = 2\, h\left(\frac{N+1}{2} - k\right)$, $k = 1, 2, \ldots, N/2$.

D. All Pass Filters

An allpass filter has a frequency response with a constant magnitude as given by

$$|H(e^{j\omega})| = 1$$

This unit magnitude constraint restricts the poles and zeros of a rational system function to occur in conjugate reciprocal pairs:

$$H[z] = \prod_{k=1}^{N} \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}$$

Thus, if $H[z]$ has a pole at $z = a_k$, $H[z]$ must have a zero at the conjugate reciprocal location $z = 1/a_k^*$. If $h[n]$ is real-valued, the complex roots in $H[z]$ occur in conjugate pairs, and if these conjugate pair are combined to form second-order factors, the system function may be written as

$$H[z] = \prod_{k=1}^{N_1} \frac{z^{-1} - b_k}{1 - b_k z^{-1}} \prod_{k=1}^{N_2} \frac{d_k - c_k z^{-1} + z^{-2}}{1 - c_k z^{-1} + d_k z^{-2}} ,$$

where the coefficients b_k, c_k and d_k are real. If an allpass filter $H[z]$ is stable and causal, the poles of $H[z]$ lie inside the unit circle, $|a_k| < 1$. In the figure below, we show a pole-zero plot for an all-pass filter.

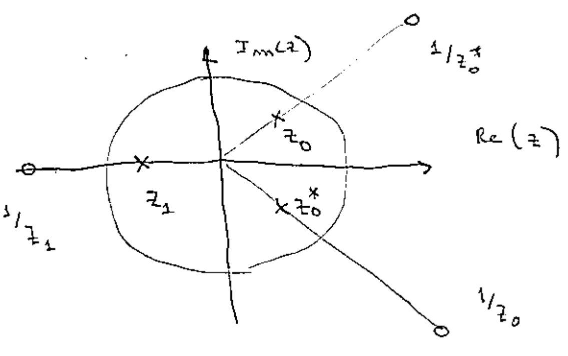


Fig. 7. Illustration of the conjugate reciprocal symmetry constraint that is placed on the poles and zeros of an all-pass system.

Allpass filters are useful for group delay equalization to compensate for phase nonlinearities.

A stable all pass filter has a group delay that is nonnegative for all ω. This follows from the fact that, for a first-order all pass factor of the form

$$H[z] = \frac{z^{-1} - \alpha^*}{1 - \alpha z^{-1}},$$

where $\alpha = r e^{j\theta}$, the group delay is

$$\tau(\omega) = \frac{1 - r^2}{|1 - r e^{j\theta} e^{-j\omega}|^2}$$

Therefore, with $0 \le r < 1$, it follows that $\tau(\omega) > 0$. Because a general all pass filter has a group delay that is a sum of terms of this form, the group delay of a rational, stable, and causal allpass filter is nonnegative.

A filter may be cascaded with an allpass filter without changing the magnitude of the frequency response. If the pole of the allpass filter cancels a zero, the zero is replaced with one at the conjugate reciprocal location. Thus, flipping one or more zeros of the system function about the unit circle does not change the magnitude of the frequency response.

Ex For a filter with system function $H[z] = \dfrac{1 - 0.2z^{-1}}{1 - 0.5z^{-1}}$

the magnitude of the frequency response will not be changed if it is cascaded with the allpass filter

$$H_{ap}[z] = \frac{z^{-1} - 0.2}{1 - 0.2z^{-1}}$$

This allpass filter flips the zero at $z = 0.2$ in $H[z]$ to its reciprocal location, $z = 5$, and the new filter has a system function

$$G[z] = \frac{z^{-1} - 0.2}{1 - 0.5z^{-1}}$$

E. Minimum Phase Systems

A stable and causal LTI system with a rational system function of the form given by

$$H[z] = \frac{\sum_{k=0}^{q} b[k]\, z^{-k}}{1 + \sum_{k=1}^{p} a[k] . z^{-k}} = A \frac{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}{\prod_{k=1}^{p} (1 - \alpha_k z^{-1})} ,$$

where all the poles are inside the unit circle, $|\alpha_k| < 1$. The zeros, however, may lie anywhere in the z-plane. In some applications, it is necessary to constrain a system so that its inverse, $G[z] = 1/H[z]$, is also stable and causal. This requires that the zeros of $H[z]$ lie inside the unit circle, $|\beta_k| < 1$. A stable and causal filter that has a stable and causal inverse is said to have minimum phase.

Definition : A rational system function with all its poles and zeros inside the unit circle is said to have minimum phase.

A minimum phase system is uniquely defined by the magnitude of its Fourier transform, $|H(e^{j\omega})|$. The procedure to find $H[z]$ from $|H(e^{j\omega})|$ is as follows. Given $|H(e^{j\omega})|$, we find $|H(e^{j\omega})|^2$, which is a function of $\cos(k\omega)$. Then, by replacing $\cos(k\omega)$ with $\frac{1}{2}(z^k + z^{-k})$, we have

$$G[z] = H[z] . H[z^{-1}]$$

Finally, the minimum phase system is then formed from the poles and zeros of $G[z]$ that are inside the unit circle.

Ex : Let $H[z]$ be a minimum phase system with a Fourier transform magnitude

$$|H(e^{j\omega})|^2 = \frac{17}{16} - \frac{1}{2} \cos \omega$$

Expressing $\cos w$ in terms of complex exponentials

$$|H(e^{jw})|^2 = \frac{17}{16} - \frac{1}{4} e^{jw} - \frac{1}{4} e^{-jw}$$

and replacing e^{jw} with z and e^{-jw} with z^{-1}, we have

$$G[z] = H[z] . H[z^{-1}] = \frac{17}{16} - \frac{1}{4} z - \frac{1}{4} z^{-1} = \left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{1}{4} z\right)$$

Thus, the minimum phase system is

$$H[z] = 1 - \frac{1}{4} z^{-1}$$

A stable and causal system may always be factored into a product of a minimum phase system with an allpass system:

$$H[z] = H_{min}[z] . H_{ap}[z]$$

The procedure for performing this factorisation is as follows. First, all of the zeros of $H(z]$ that are outside the unit circle are reflected inside the unit circle to their conjugate reciprocal location. The resulting system function is minimum phase, $H_{min}[z]$. Then, the allpass filter is selected so that it reflects the appropriate set of zeros of $H_{min}(z)$ back outside the unit circle.

Ex: For the system function

$$H[z) = \frac{1 - 2z^{-1}}{(1 - 0.2z^{-1})(1 - 0.7z^{-1})}$$

The minimum phase factor is

$$H[z] = \frac{z^{-1} - 2}{(1 - 0.2z^{-1})(1 - 0.7z^{-1})}$$

Then, to reflect the zero at $z = 0.5$ back outside the unit circle to $z = 2$, we use the allpass factor

$$H_{ap}[z] = \frac{1 - 2z^{-1}}{z^{-1} - 2}$$

Properties of Minimum Phase Systems:

1) All systems that have the same Fourier transform magnitude, the minimum phase system has the minimum group delay. Specifically, let $H_{min}[z]$ be a minimum phase system and let $H[z]$ be another system with the same magnitude. The group delay for $H[z]$ may be written as

$$\tau_h(\omega) = \tau_{min}(\omega) + \tau_{ap}(\omega),$$

where $\tau_{ap}(\omega)$ is the group delay of a stable and causal allpass system. Because $\tau_{ap}(\omega) > 0$, the group delay of $H[z]$ will be larger than the group delay of the minimum phase system $H_{min}[z]$. Furthermore, because the phase is the negative of the integral of the group delay, the minimum phase system is also said to have the minimum phase-lag.

2) Minimum phase systems have the minimum energy delay. In particular, if $h_{min}[n]$ is the impulse response of a minimum phase system, and $h[n]$ is the impulse response of another causal system that has the same magnitude response

$$\sum_{k=0}^{n} |h[k]|^2 < \sum_{k=0}^{n} |h_{min}[k]|^2 \quad , \text{ for any } n \geq 0.$$

F. Simple Digital Filters

In this section, we describe several low-order FIR and IIR digital filters with frequency responses that are often satisfactory in a number of applications.

i) Low pass FIR digital filters

The simplest lowpass filter is the moving-average filter given by

$$h[n] = \begin{cases} \frac{1}{M} & , \quad 0 \le n \le M-1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

When $M = 2$, the system function is described by

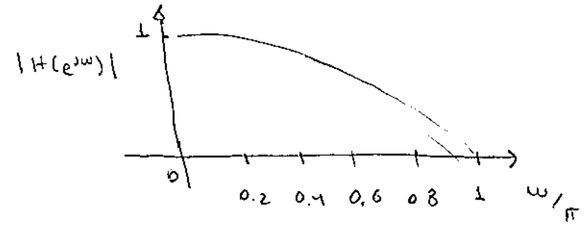
$$H(z) = \frac{1}{2}\left(1 + z^{-1}\right) = \frac{z+1}{2z}.$$

The above transfer function has a zero at $z=-1$ and a pole at $z=0$. As w increases from 0 to π, the magnitude $|H(z=e^{jw})|$ decreases from 1 to zero. This function monotonically decreases from $w=0$ to $w=\pi$. In fact, the frequency response of this filter is

$$H(e^{jw}) = \frac{e^{jw}+1}{2e^{jw}} = e^{-jw/2}\,\frac{\left(e^{jw/2}+e^{-jw/2}\right)}{2}$$

$$= e^{-jw/2}\cos\left(\frac{w}{2}\right),$$

where its frequency response is shown below



$|H(e^{j0})| = 1$

and

$|H(e^{j\pi})| = 0$

The frequency $w = w_c$ at which $|H(e^{jw_c})| = \frac{1}{\sqrt{2}}|H(e^{j0})| = 1$ is of practical interest since the gain in dB is given by

$$G(w_c) = 20\log_{10}|H(e^{jw_c})| = \underbrace{20\log_{10}|H(e^{j0})|}_{0} - 20\log_{10}\sqrt{2}$$

$$\simeq -3\text{ dB},$$

where w_c is called the 3-dB cutoff frequency.

In order to determine the expression for ω_c we set

$$|H(e^{j\omega_c})|^2 = \cos^2(\omega_c/2) = 1/2$$

which yields

$$\omega_c = \pi/2 .$$

The 3 dB cutoff frequency can be considered as the passband edge frequency. As a result, this filter has a passband width of approximately $\pi/2$, whereas the stopband is from $\pi/2$ to π. Signal with rapid fluctuations in sample values are associated with high frequency components that are eliminated by a moving-average filter of this type. This approach results in smoother waveforms

ii) High pass FIR digital filters

The simplest highpass FIR filter is obtained by replacing z with $-z$ in $H[z] = \dfrac{z+1}{2z}$, resulting in

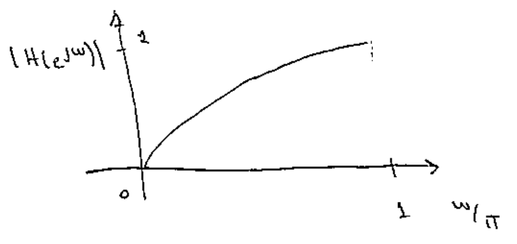
$$H[z] = \frac{1 - z^{-1}}{2}$$

The corresponding frequency response is given by

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega}}{2} = j\, e^{-j\omega/2} \left(\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} \right)$$

$$= j\, e^{-j\omega/2} . \sin(\omega/2),$$

whose magnitude response is shown below



$|H(e^{j0})| = 0$

and

$|H(e^{j\pi})| = 1$

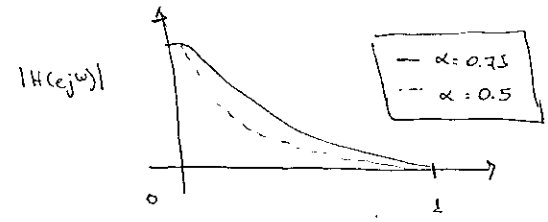
This transfer function has a 3 dB cutoff frequency at $\pi/2$. and has a zero at $\omega = 0$, which is in the stopband (0 to $\pi/2$) of the filter. An application of such filters is in moving-target-indicator (MTI) radars. In these radars, interfering signals called clutter are generated from objects in the path of the radar beam. The clutter generated by ground echoes and radar returns has frequency components near zero frequency (dc) and can be removed by highpass filtering.

iii) Lowpass IIR Digital filters

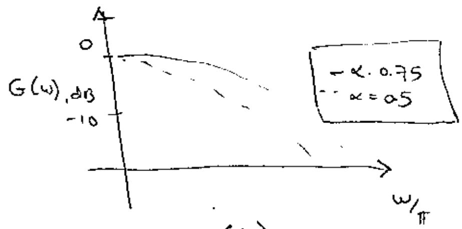
A first-order lowpass IIR digital filter has a transfer function given by

$$H[z] = \frac{1-\alpha}{2} \; \frac{1+z^{-1}}{1+\alpha z^{-1}},$$

where $|\alpha| < 1$ for stability ; there is a zero at $z = -1$ ($\omega = \pi$ or stopband of the filter) and a pole at $z = \alpha$. The magnitude response of the filter decreases from 1 to 0 as ω goes from 0 to π, as illustrated in the figure below



(a)



(b)

$|H(e^{j0})| = 1$ and $|H(e^{j\pi})| = 0$

The frequency response is given by

$$H(e^{j\omega}) = \frac{1-\alpha}{2} \; \frac{1+e^{-j\omega}}{1+\alpha e^{-j\omega}}$$

In order to determine the 3 dB cutoff frequency ω_c we set $|H(e^{j\omega_c})|^2 = 1/2$ and arrive at

$$\frac{(1-\alpha)^2 (1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)} = \frac{1}{2} \quad \text{or} \quad (1-\alpha)^2(1+\cos\omega_c) = 1+\alpha^2-2\alpha\cos\omega_c,$$

whose solution is $\boxed{\cos\omega_c = \frac{2\alpha}{1+\alpha^2}}$ with $\boxed{\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}}$

iv) Highpass IIR Digital Filters

A first-order highpass system function is given by

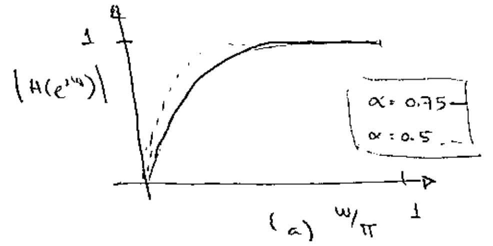
$$H[z] = \frac{1+\alpha}{2} \; \frac{1-z^{-1}}{1-\alpha z^{-1}},$$

where $|\alpha| < 1$ for stability.

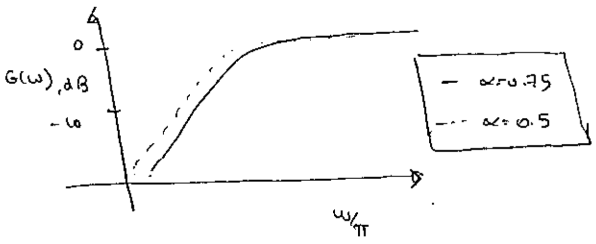
The frequency response is given by

$$H(e^{j\omega}) = \frac{1+\alpha}{2} \quad \frac{1 - e^{-j\omega}}{1 - \alpha e^{-j\omega}},$$

where its illustration is shown below



(a)



The 3dB cutoff frequency w_c can be obtained by setting

$|H(e^{j\omega_c})|^2 = \frac{1}{2}$ which yields

$$\frac{(1+\alpha)^2 \, (1-e^{-j\omega})(1-e^{j\omega})}{2^2 \, (1-\alpha e^{-j\omega})(1-\alpha e^{j\omega})} = \frac{1}{2}$$

$$\frac{(1+\alpha)^2 \, (1 - \cos w_c)}{2 \, (1+\alpha^2 - 2\alpha \cos w_c)} = \frac{1}{2}$$

$$(1+\alpha)^2 \, (1 - \cos w_c) = 1 + \alpha^2 - 2\alpha \cos w_c$$

The solution is given by

$$\cos w_c = \frac{2\alpha}{1+\alpha^2} \qquad \text{with} \qquad \alpha = \frac{1 - \sin w_c}{\cos w_c}$$

Ex: Design a 1^{st}-order highpass filter with a 3dB cutoff frequency of 0.8π.

$$\left.\begin{aligned} \sin(w_c) &= \sin(0.8\pi) = 0.58 \\ \cos(w_c) &= \cos(0.8\pi) = -0.81 \end{aligned}\right\} \quad \alpha = -0.51$$

Therefore, the transfer function of $H(z)$ is

$$H(z) = 0.24 \left(\frac{1 - z^{-1}}{1 + 0.51 z^{-1}} \right)$$

v) Bandpass IIR Digital Filters

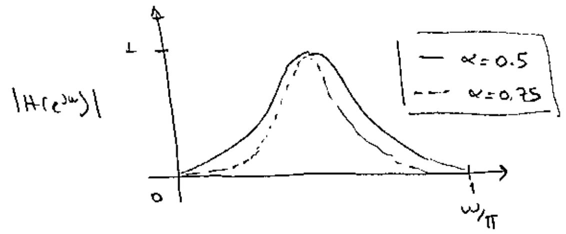
A second-order bandpass digital filter is described by

$$H[z] = \frac{1-\alpha}{2} \frac{1 - z^{-2}}{1 - \beta(1+\alpha) z^{-1} + \alpha z^{-2}}$$

The frequency response is given by

$$H(e^{j\omega}) = \frac{1-\alpha}{2} \frac{1 - e^{-j\omega 2}}{1 - \beta(1+\alpha) e^{-j\omega} + \alpha e^{-j\omega 2}}$$

and shown below



$|H(e^{j0})| = 0$ and $|H(e^{j\pi})| = 0$ $\quad |H(e^{j\omega_0})| = 1$

The square magnitude function is given by

$$|H(e^{j\omega})|^2 = \frac{(1-\alpha)^2 (1 - \cos 2\omega)}{2\left[1 + \beta^2(1+\alpha)^2 + \alpha^2 - 2\beta(1+\alpha)^2 \cos\omega + 2\alpha \cos 2\omega\right]}$$

The centre frequency of the bandpass filter is associated with the maximum value of unity at $\omega = \omega_0$ as given by

$$\omega_0 = \cos^{-1}(\beta).$$

The frequencies ω_{c1} and ω_{c2} correspond to those where $|H(e^{j\omega})|^2$ is equal to $1/2$ are called the 3dB cutoff frequencies and their difference, B_ω, assuming $\omega_{c2} > \omega_{c1}$ is the 3dB bandwidth:

$$B_\omega = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

Ex: Design a 2nd order bandpass filter with $\omega_0 = 0.4\pi$ and $B_\omega = 0.1\pi$

$$\left.\begin{aligned} \beta &= \cos(\omega_0) = \cos(0.4\pi) = 0.31 \\ \frac{2\alpha}{1+\alpha^2} &= \cos(B_\omega) = \cos(0.1\pi) = 0.95 \end{aligned}\right\} \quad \alpha_1 = 1.37 \text{ and } \alpha_2 = 0.72$$

$$H_1[z] = -0.18 \frac{1 - z^{-2}}{1 - 0.73z^{-1} + 1.37z^{-2}} \quad \text{and} \quad H_2[z] = 0.13 \frac{1 - z^{-2}}{1 - 0.53z^{-1} + 0.72z^{-2}}$$

G. Digital Filter Structures

In this section, we consider ways to implement LTI discrete-time systems. In particular, we look at the representation of IIR and FIR system functions in block diagram and signal flowgraph forms. Such structural representations using interconnected basic building blocks is the first step towards the hardware or software implementation of an LTI digital filter.

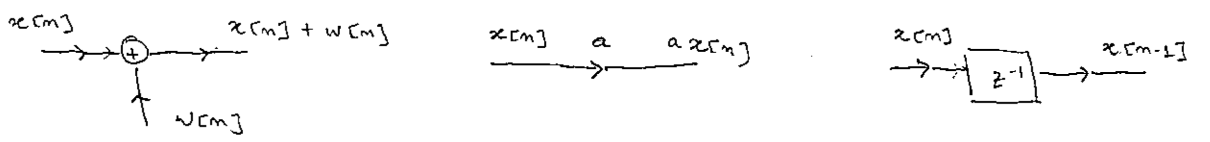
The input-output relations of an LTI digital filter can be expressed in the time-domain by the convolution sum:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

or by the difference equation given by

$$y[n] = \sum_{k=0}^{q} b[k] \cdot x[n-k] - \sum_{k=1}^{p} a[k] \cdot y[n-k].$$

The basic computational elements required to compute $y[n]$ are adders, multipliers and delays. It is often convenient to use a block diagram to illustrate how these adders, multipliers and delays are interconnected to implement a system. The notation used for these elements is shown in the figure below.



(a) Adder (b) Multiplier (c) A unit delay

Fig. Notation used for basic elements

An alternative representation of an LTI system can be obtained in the form of a signal flowgraph. Each branch has an input and an output with the direction indicated by an arrowhead.

Adders correspond to nodes with more than one incoming branch, and branch points are nodes with more than one outgoing branch, as illustrated below.

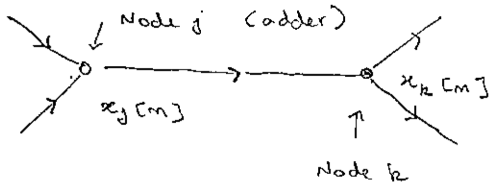


Fig. Signal flowgraph consisting of nodes, branches and node variables.

With a linear flowgraph, the output of each branch is a linear transformation of the branch input, and the linear operator is indicated next to the arrow. For LTI digital filters, these operators consist of multiplies and delays. There are two types of nodes:

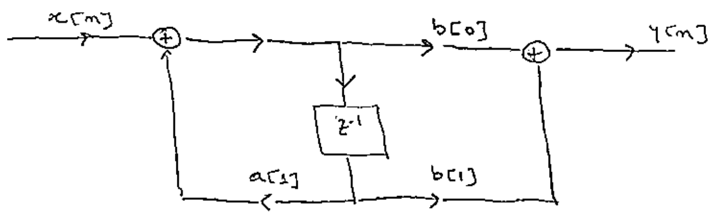
i) Source nodes: They are used for sequences that are input to the filter as they have no incoming branches;

ii) Sink nodes: these are nodes that have only entering branches and are used to represent output sequences.

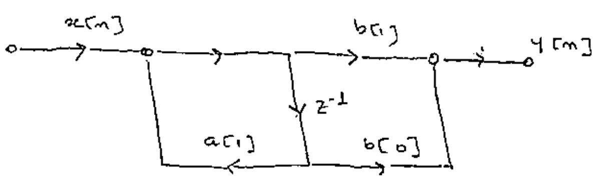
Ex: Consider the first-order discrete-time system described by

$$y[n] = b[0]\,x[n] + b[1]\,x[n-1] + a[1]\,y[n-1]$$

Draw the block diagram of the system



Using a signal flowgraph, this system is represented as



An important goal of designers is to develop realizations of system functions that are suitable for implementation. When we analyze digital filters under infinite precision arithmetic any realization behaves identically to the other equivalent structures. However, in practice due to the finite wordlength limitations, different realizations behave differently. Hence, it is important to choose a structure that has the least quantisation effects on the implementation and behaviour of the digital filter. It is also important to consider the implementation complexity of the realization of the filter in terms of the total number of multipliers and adders required.

H. Structures for FIR systems

In this section, we consider the realization of FIR digital filters, which is characterized by

$$H[z] = \sum_{k=0}^{N} h[k] \cdot z^{-k},$$

where N is the order of the filter and H(z) is a polynomial in z^{-1} of degree N.

In the time domain, the input-output relationship of the above FIR filter is described by

$$y[n] = \sum_{k=0}^{N} h[k]\, x[n-k],$$

where y[n] and x[n] are the output and input sequences, respectively.

FIR filters are preferred in many applications because they can be designed to provide exact linear phase over the whole frequency range and are always BIBO stable independent of the filter coefficients. For computing each value of y[n], we need N+1 multiplications and N additions.

i) Direct form

Structures in which the multiplier coefficients are precisely the coefficients of the transfer function are called direct form structures. They are the most common way to implement an FIR filter and is often realized using a tapped delay line as shown in the figure below.

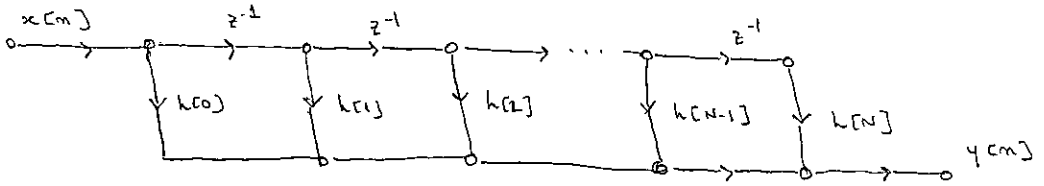


Fig. Tapped delay line or transversal structure

This structure requires N + 1 multiplications, N additions and N delays. However, if there are some symmetries in the impulse response, it may be possible to reduce the number of multiplications.

ii) Cascade Form

A higher-order FIR system function can also be realized as a cascade of FIR sections with each section characterized by either a first-order or a second-order system function. To this end, the system function can be factored into a product of first-order factors

$$H[z] = \sum_{n=0}^{N} h[n] z^{-n} = A \prod_{k=1}^{N} (1 - \alpha_k z^{-1}),$$

where α_k for $k = 1, \ldots, N$ are the zeros of $H[z]$. If $h[n]$ is real, the complex roots of $H[z]$ occur in complex conjugate pairs which can be combined to form second-order factors with real coefficients:

$$H[z] = A \prod_{k=1}^{N_s} \left(1 + b_k[1] z^{-1} + b_k[2] z^{-2} \right),$$

which can be implemented as the cascade of of second-order FIR filters as shown next.

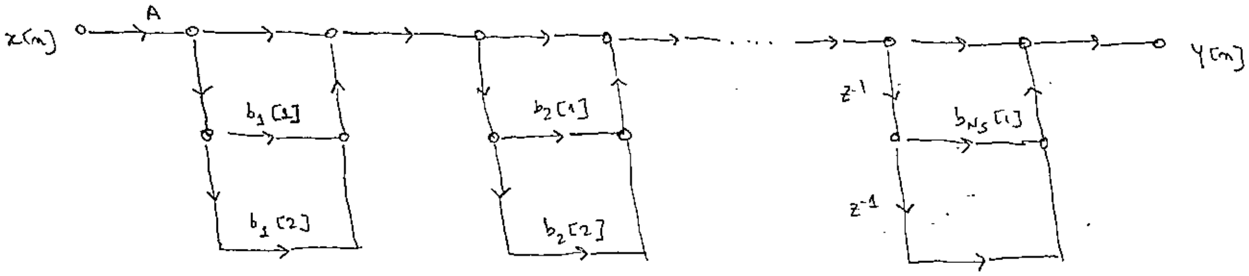


Fig. An FIR filter implemented as a cascade of second-order systems.

It should be remarked that each second-order stage can also be realized in the transposed direct form. The cascade form is canonic and employs N+1 multipliers and N adders to implement an N-th order FIR system function.

iii) Linear Phase Filters

Linear phase filters have an impulse response that is either symmetric as given by

$$h[n] = h[N-n] \; ,$$

or anti-symmetric as expressed by

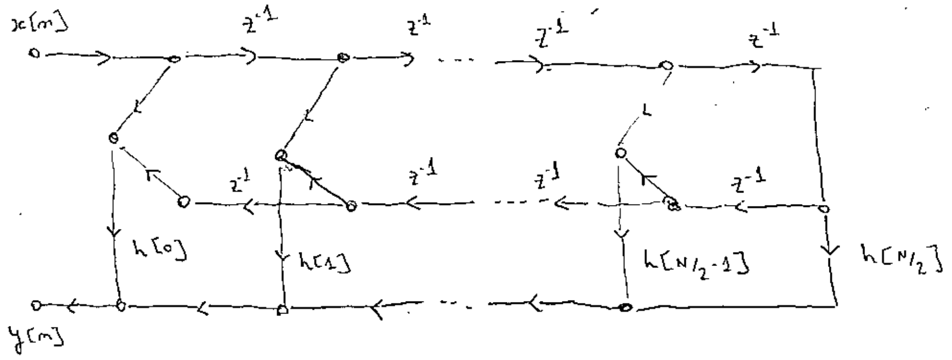
$$h[n] = -h[N-n] \; .$$

This simmetry may be exploited to simplify the network structure. For example, if N is even and h[n] is symmetric (type I filter), we have

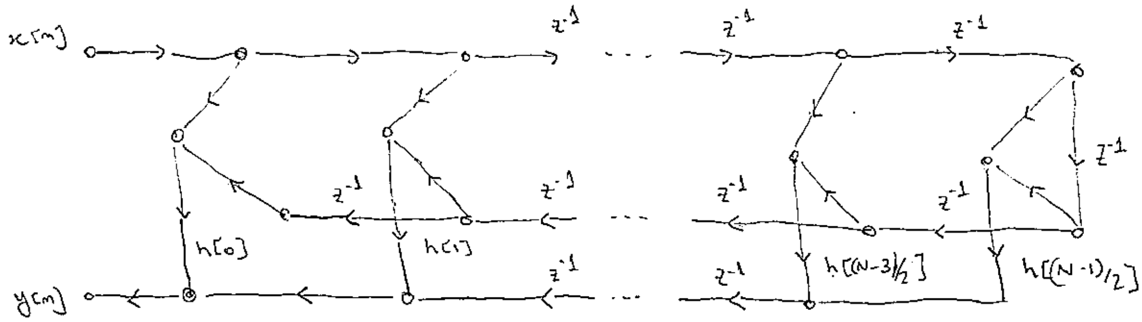
$$y[n] = \sum_{k=0}^{N} h[k] \cdot x[n-k]$$

$$= \sum_{k=0}^{N/2-1} h[k] \cdot \left(x[n-k] + x[n-N+k] \right) + h[N/2]\, x\left[n - \frac{N}{2}\right]$$

Therefore, forming the sums $(x[n-k] + x[n-N+k])$ prior to multiplying by h[k] reduces the number of multiplications. The resulting structures are shown in the next figure



(a)



(b)

Fig. Direct form implementation for linear phase filters (a) Type I and (b) Type II.

I. Structures for IIR Systems

The input - output relationship of a causal IIR filter can be described by the rational system function given by

$$H[z] = \frac{B[z]}{A[z]} = \frac{\sum_{k=0}^{q} b[k] \cdot z^{-k}}{1 + \sum_{k=1}^{p} a[k] \cdot z^{-k}},$$

and also by difference equation expressed by

$$y[n] = \sum_{k=0}^{q} b[k] \cdot x[n-k] - \sum_{k=1}^{p} a[k] \cdot y[n-k]$$

In what follows, we describe several implementation strategies for this system.

i) Direct Forms

There are two direct form structures referred to as direct form I and direct form II. The direct form I structure is an implementation that results when the difference equation is written as a pair of equations described by

$$w[n] = \sum_{k=0}^{q} b[k] \cdot x[n-k]$$

$$y[n] = w[n] - \sum_{k=1}^{p} a[k]\, y[n-k]$$

The first equation corresponds to an FIR filter with input $x[n]$ and output $w[n]$, whereas the second equation corresponds to an all-pole filter with input $w[n]$ and output $y[n]$. Therefore, this pair of equations represents a cascade of two systems given by

$$Y[z] = \frac{1}{A[z]} B[z] \cdot X[z] ,$$

which is shown below.

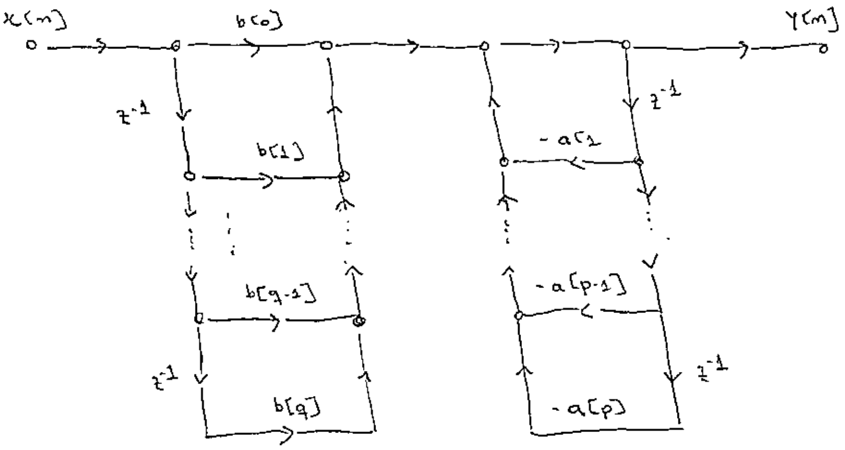


Fig. Direct form I realization of an IIR filter.

The direct form I structure requires $p+q$ delays and $p+q+1$ multiplications and $p+q$ additions per output sample.

The direct form II structure is obtained by reversing the order of the cascade of $B[z]$ and $\frac{1}{A[z]}$ as illustrated in the figure below.

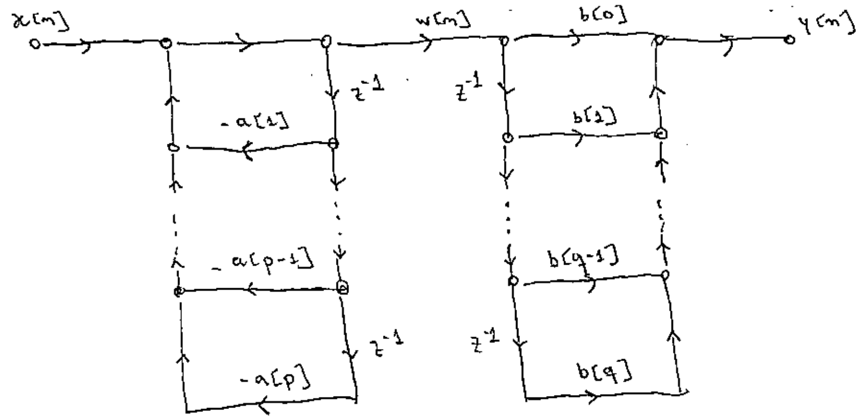


Fig. Reversing the order of the cascade in the direct form I filter structure.

With this implementation, $x[n]$ is first filtered with the all-pole filter $1/A[z]$ and then with $B[z]$: as described by

$$Y[z] = B[z] \cdot \left[\frac{1}{A[z]} X[z] \right]$$

If we denote the output of the all-pole filter $1/A[z]$ by $w[n]$, this structure is described by the following pair of difference equations:

$$w[n] = x[n] - \sum_{k=1}^{p} a[k]\, w[n-k]$$

$$y[n] = \sum_{k=0}^{q} b[k] \cdot w[n-k]$$

This structure can be simplified by taking into account that the two sets of delays are delaying the same sequence, as shown below.

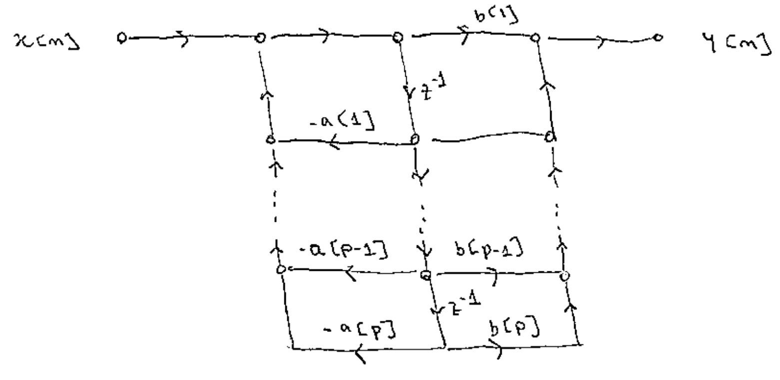


Fig. Direct form II realization of an IIR filter with $p=q$

The computational requirements for a direct form II structure is the maximum of p and q delays, i.e., max (p,q), p+q+1 multiplications and p+q additions per output sample. The direct form II structure is said to be canonic because it uses the minimum number of delays for a given H[z].

ii) Cascade Structure

The cascade structure is derived by factoring the numerator and denominator polynomials of H[z]: as expressed by

$$H[z] = \frac{\sum_{k=0}^{q} b[k] \cdot z^{-k}}{1 + \sum_{k=1}^{p} a[k] \cdot z^{-k}} = A \prod_{k=1}^{\max(p,q)} \frac{1 - \beta_k z^{-1}}{1 - \alpha_k z^{-1}}$$

This factorization corresponds to a cascade of first-order filters, each having one pole and one zero. In general, the coefficients α_k and β_k will be complex. However, if h[n] is real the roots of H[z] will occur in complex conjugate pairs, and these complex conjugate factors may be combined to form second-order factors with real coefficients:

$$H_k[z] = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

There is considerable flexibility in how a system can be implemented in cascade form. For example, there are different pairings of the poles and zeros and different ways in which the sections may be ordered. An example of a fourth-order IIR filter implemented as a cascade of two second-order systems in direct form II is shown next.

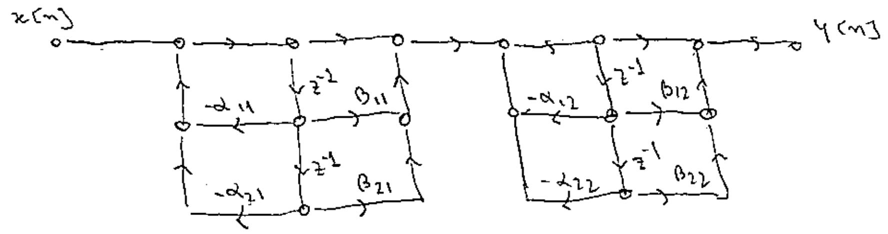


Fig. A fourth-order filter implemented as a cascade of two direct form II second-order systems.

iii) Parallel Structure

An alternative to factoring $H[z]$ is to expand the system function using a partial fraction expansion. Consider the system function

$$H[z] = \frac{\sum_{k=0}^{q} b[k] \cdot z^{-k}}{1 + \sum_{k=1}^{p} a[k] \cdot z^{-k}} = A \frac{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}{\prod_{k=1}^{p} (1 - \alpha_k z^{-1})}$$

If $p > q$ and $\alpha_i \neq \alpha_k$ (distinct roots), $H[z]$ may be expanded as a sum of p first-order factors as follows:

$$H[z] = \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}},$$

where the coefficients A_k and α_k are in general complex. This expansion corresponds to a sum of p first-order system functions and may be realized by connecting these systems in parallel. If $h[n]$ is real, the poles of $H[z]$ appear in complex conjugate pairs, and these complex roots in the partial fraction expansion may be combined to form second-order systems with real coefficients:

$$H[z] = \sum_{k=1}^{N_s} \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

A sixth-order filter implemented as a parallel connection of three second-order direct form II systems is shown below.

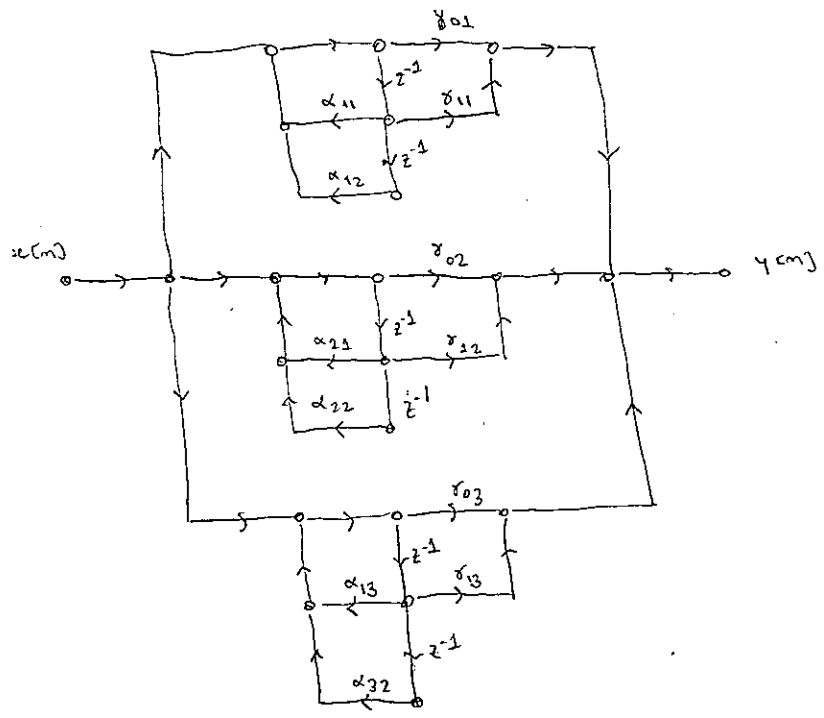


Fig. A sixth-order IIR filter implemented as a parallelized connection of three second-order direct form II structures.

If $p \leq q$, the partial fraction expansion will also contain a term of the form

$$c_0 + c_1 z^{-1} + \dots + c_{q-p} z^{-(q-p)},$$

which is an FIR filter that is placed in parallel with the other terms in the expansion of $H[z]$.

I. Computational Complexity of Filter Structures.

The computational complexity of digital filter structures is measured in terms of multiplications and additions, which are shown in the tables below.

Table 1 : Complexity of FIR filters

Structure	No of multipliers	No of two-input adders
Direct form	$N+1$	N
Cascade form	$N+1$	N
Linear phase	$\left\lfloor \frac{N+2}{2} \right\rfloor$	N

Table 2 : Complexity of IIR filters

Structure	No. of multipliers	No. of two-input adders
Direct form I and II	$2N+1$	$2N$
Cascade form	$2N+1$	$2N$
Parallel form	$2N+1$	$2N$

This is equivalent to

$$\sum_{n=-\infty}^{\infty} |h[n]| \, z^{-n} < \infty$$

for $|z| = 1$, the ROC of the system function must include the unit circle if the system is stable

" Because the impulse response of a causal system is right-sided, we have $h[n] = 0$ for $n < 0$. The ROC of $H[z]$ will be the exterior of a circle, $|z| > \alpha$. Because no poles may lie within the ROC, all the poles of $H[z]$ must lie on or inside the circle $|z| \leq \alpha$. Causality imposes tight constraints on LTI systems and the first of these is the Paley-Wiener theorem.

Paley-Wiener Theorem: If $h[n]$ has finite energy and $h[n] = 0$ for $n < 0$,

$$\int_{-\pi}^{\pi} \left| \ln |H(e^{j\omega})| \right| d\omega < \infty$$

One of the consequences of this theorem is that the frequency response of a stable and causal system cannot be zero over any finite band of frequencies. Therefore, any stable ideal frequency selective filter will be non causal.

Causality also places restrictions on the real and imaginary parts of the frequency response. For example, if $h[n]$ is real, $h[n]$ may be decomposed into its even and odd parts:

$$h[n] = h_e[n] + h_o[n],$$

where $h_e[n] = \frac{1}{2}\left[h[n] + h[-n]\right]$ and $h_o[n] = \frac{1}{2}\left[h[n] - h[-n]\right]$.

If $h[n]$ is causal, it is uniquely defined by its even part:

$$h[n] = 2 h_e[n] u[n] - h_e[n] \cdot \delta[n]$$

If $h[n]$ is absolutely summable, the DTFT of $h[n]$ exists, and $H[e^{j\omega}]$ may be written in terms of its real and imaginary parts:

$$H(e^{j\omega}) = H_R[e^{j\omega}] + j \, H_I(e^{j\omega}]$$

Therefore, because $H_R[e^{j\omega}]$ is the DTFT of the even part of $h[n]$, it follows that if $h[n]$ is real, stable and causal, $H[e^{j\omega}]$ is uniquely defined by its real part. This implies a relationship between the real and imaginary parts of $H[e^{j\omega}]$, which is given by

$$H_I[e^{j\omega}] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_R[e^{j\theta}] \cot\left(\frac{\omega-\theta}{2}\right) d\theta$$

This integral is called a discrete Hilbert transform. Specifically, $H_I(e^{j\omega}]$ is the discrete Hilbert transform of $H_R[e^{j\omega}]$.

A realizable system is one that is both stable and causal. A realizable system will have a system function with a ROC of the form $|z| > \alpha$ where $0 \leq \alpha < 1$. Therefore, any poles of $H[z]$ must lie inside the unit circle. For example, the first-order system

$$H[z] = \frac{b[0]}{1 + a[1]z^{-1}}, \quad |z| > |a[1]|$$

will be realizable (stable and causal) if and only if

$$|a[1]| < 1$$

For the second-order system $H[z] = \dfrac{b[0]}{1 + a[1]z^{-1} + a[2]z^{-2}}$, there are two zeros at the origin and poles at

$$\alpha_1, \alpha_2 = -\frac{a[1]}{2} \pm \sqrt{\frac{a^2[1] - 4a[2]}{4}}$$

These roots satisfy the following two equations:

$$a[1] = -(\alpha_1 + \alpha_2)$$
$$a[2] = \alpha_1 \cdot \alpha_2$$

From these equations, it follows that the roots of $H[z]$ will be inside the unit circle if and only if

$$|a[2]| < 1$$
$$|a[1]| < 1 + a[2]$$

The above constraints define a stability triangle in the coefficient plane as illustrated in the next figure.

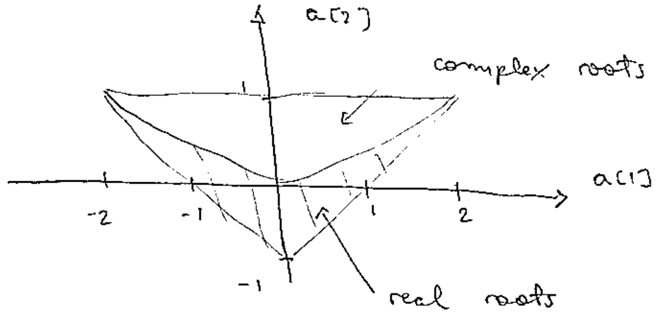


Fig. 1 The stability triangle defining the areas corresponding to real and complex roots.

Fig. 1, shows that a causal second-order system will be stable if and only if the coefficients $a[1]$ and $a[2]$ lie inside this triangle. This result is important because second-order systems are the basic building blocks for higher-order systems.

ii) Inverse Systems

For an LTI system with a system function $H[z]$, the inverse system is defined to be the system that has a system function $G[z]$ such that

$$H[z] \cdot G[z] = 1$$

In other words, the cascade of $H[z]$ and $G[z]$ produces the identity system. In terms of $H[z]$, the inverse is simply

$$G[z] = \frac{1}{H[z]}$$

For example, if $H[z]$ is a rational function of z, the inverse system is described by

$$G[z] = A^{-1} \frac{\prod_{k=1}^{p} (1 - \alpha_k z^{-1})}{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}$$

Thus, the poles of $H[z]$ become the zeros of $G[z]$ and vice-versa. The ROC associated with the inverse system is determined by the requirement that $H[z]$ and $G[z]$ have overlapping ROCs.

Ex: If $H[z] = \dfrac{1 - 0.5 z^{-1}}{1 - 0.8 z^{-1}}$, $|z| > 0.8$ then the inverse system is $G[z] = \dfrac{1 - 0.8 z^{-1}}{1 - 0.5 z^{-1}}$.

There are two possible ROCs for $g[n]$. The first is $|z| > \frac{1}{2}$ and the second is $|z| < \frac{1}{2}$. Because $|z| < \frac{1}{2}$ does not overlap the ROC of $H[z]$, the only possibility for the inverse system is $|z| > \frac{1}{2}$. In this case, the impulse response is

$$g[n] = \left(\frac{1}{2}\right)^n u[n] - 0.8 \left(\frac{1}{2}\right)^{n-1} u[n-1],$$

which is stable and causal

Suppose ~~that~~ $H[z] = \dfrac{0.5 - z^{-1}}{1 - 0.8 z^{-1}}$, $|z| > 0.8$

In this case, we have $G[z] = \dfrac{1 - 0.8 z^{-1}}{0.5 - z^{-1}}$ where the ROC can be given by $|z| > 2$ or $|z| < 2$.

Because both $|z| > 2$ and $|z| < 2$ overlap the ROC, both are valid inverse systems.

The first, which has a ROC $|z| > 2$, has an impulse response given by

$$g[n] = 2 (2)^n u[n] - 1.6 (2)^{n-1} u[n-1],$$

which is causal but unstable.

The second, with a ROC given by $|z| < 2$, has an impulse response given by

$$g[n] = -2 (2)^n u[-n-1] + 1.6 (2)^{n-1} u[-n],$$

which is stable but noncausal.

iii) Impulse Response for Rational System Functions

Consider the rational system function of an LTI system given by

$$H[z] = A \frac{\prod_{k=1}^{q} (1 - \beta_k z^{-1})}{\prod_{k=1}^{p} (1 - \alpha_k z^{-1})}$$

Assuming first-order poles with $\alpha_k \neq \beta_k$, $H[z]$ may be expanded using a partial fraction expansion:

$$H[z] = \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}}$$

If the system is causal, the impulse response is

$$h[n] = \sum_{k=1}^{p} A_k \alpha_k^n u[n]$$

When $p \leq q$, the partial fraction expansion has the form

$$H[z] = \sum_{k=0}^{q-p} B_k z^{-k} + \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}}$$

If the system is causal, the impulse response becomes

$$h[n] = \sum_{k=0}^{q-p} B_k \delta[n-k] + \sum_{k=1}^{p} A_k \alpha_k^n u[n]$$

If $p = 0$, $H[z]$ has only zeros and is described by

$$H[z] = \sum_{k=0}^{q} b[k] z^{-k}$$

and $h[n]$ is finite in length with

$$h[n] = \sum_{k=0}^{q} b[k] \delta[n-k]$$

These systems are called finite-impulse response (FIR) filters. If $p > 0$ then $H[z]$ is infinite in length and these systems are called infinite-length impulse response (IIR) filters.

If $h[n]$ is real, $H[z] = H^*[z^*]$ and the complex poles and zeros of $H[z]$ occur in complex-conjugate pairs. For example, if $\alpha_k = r_k e^{jw_k}$ is a complex-valued pole $\alpha_k^* = r_k e^{-jw_k}$ will also be a pole.

iv) Frequency Response for Rational System Functions

The frequency response of an LTI system can be found by evaluating $H[z]$ on the unit circle. For a rational function of z, the freq. response may be found geometrically from the poles and zeros of $H[z]$. With $H[z]$ written in factored form, the freq. response is given by

$$H(e^{jw}) = A \frac{\prod_{k=1}^{q} (1 - \beta_k e^{-jw})}{\prod_{k=1}^{p} (1 - \alpha_k e^{-jw})},$$

where the magnitude of the frequency response is

$$|H(e^{jw})| = |A| \frac{\prod_{k=1}^{q} |1 - \beta_k e^{-jw}|}{\prod_{k=1}^{p} |1 - \alpha_k e^{-jw}|}$$

and the phase is given by

$$\phi_h(w) = \sum_{k=1}^{q} \arg(1 - \beta_k e^{-jw}) - \sum_{k=1}^{p} \arg(1 - \alpha_k e^{-jw})$$

In order to evaluate the frequency response geometrically, we consider the poles and zeros of the system function and a diagram shown in the figure below.

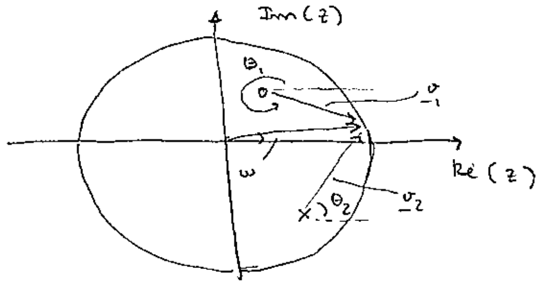


Fig 2. Evaluation of the frequency response

Each term in the numerator $|1 - \beta_k e^{-jw}|$ is the length of the vector $\underline{v}_1$ from the zero to the unit circle at $z = e^{jw}$, whereas each term in the denominator $|1 - \alpha_k e^{-jw}|$ is the length of the vector $\underline{v}_2$ from the pole $z = \alpha_k$ to the unit circle at $z = e^{jw}$.

Concerning the angles, we have $\arg(1-\beta_k e^{-j\omega}) = \Theta_1 - \omega$, where Θ_1 is the angle subtended by the vector from the zero at $z=\beta_k$ to the unit circle at $z=e^{j\omega}$. For the denominator (poles), we have $\arg(1-\alpha_k e^{-j\omega}) = \Theta_2 - \omega$, where Θ_2 is the angle of the vector from the pole at $z=\alpha_k$ to the unit circle at $z=e^{j\omega}$.

C. Systems with Linear Phase

An LTI system has linear phase if the frequency response has the form

$$H(e^{j\omega}) = |H(e^{j\omega})| \, e^{-j\omega},$$

where α is a real number. Thus, linear phase systems have a constant group delay as described by

$$\tau_h(\omega) = \alpha$$

A system with generalized linear phase has the frequency response in the form

$$H(e^{j\omega}) = A(e^{j\omega}) \, e^{-j(\alpha\omega-\beta)},$$

where $A(e^{j\omega})$ is a real-valued function of ω and β is a constant. The term linear phase is used to denote a system that has either linear phase or generalized linear phase.

Ex: Consider the FIR system with impulse response

$$h[n] = \begin{cases} 1 & n=0,1,\ldots,N \\ 0 & \text{else} \end{cases}$$

The frequency response is

$$H(e^{j\omega}) = e^{-jN\omega/2} \frac{\sin\left(\frac{N+1}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Therefore, this system has generalised linear phase with $\alpha = N/2$ and $\beta = 0$.

In order for a causal system with a rational system function to have linear phase, the impulse response must be finite in length. For this reason, IIR filters cannot have (generalised) linear phase. For an FIR filter with a real-valued impulse response of length N+1, a sufficient condition for this filter to have generalised linear phase is that the impulse response be symmetric:

$$h[n] = h[N-n],$$

which implies $\alpha = \frac{N}{2}$ and $\beta = 0$ or π. Another sufficient condition is that $h[n]$ be antisymmetric:

$$h[n] = -h[N-n],$$

which corresponds to the case in which $\alpha = \frac{N}{2}$ and $\beta = \pi/2$ or $3\pi/2$.

Linear phase filters may be classified into four types, depending on whether $h[n]$ is symmetric or anti-symmetric and whether N is even or odd. In what follows, we assume that $h[n]$ is real-valued and that $h[0]$ is the first nonzero value of $h[n]$.

Type I Linear Phase Filters:

A type I linear phase filter has a symmetric impulse response given by

$$h[n] = h[N-n], \quad 0 \leq n \leq N,$$

where N is even. The centre of symmetry is about the point $\alpha = N/2$, which is an integer as illustrated in the next figure.

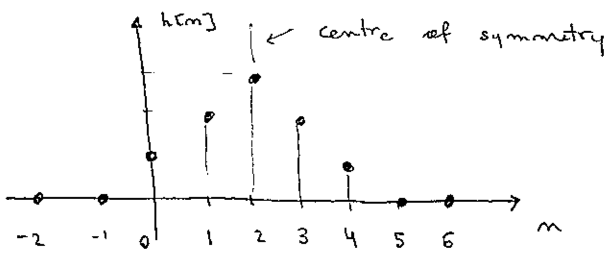


Fig. 3. Impulse response of a Type I linear phase filter.