Digital Signal Processing

Contents

VI. Design of FIR Filters

A. Window-based Designs

B. Optimisation-based Designs

C. Examples

VII. Multirate Signal Processing and Filter Banks

A. Basics, Decimation and Interpolation

B. Alteration of Sampling Rates

C. Polyphase and other Structures

D. Digital Filter Banks

E. Applications

VIII. Statistical Signal Processing

A. Review of Random variables and Stochastic Processes

B Modelling of Problems with Digital Filters

C. The Wiener Filter

D. Linear Prediction

E. Adaptive Algorithms

F. Applications

I. Discrete-time Signals and Systems

In this chapter, we begin our study of digital signal processing by reviewing the elements of a discrete-time signal and a discrete-time system. In particular, we will develop basic concepts such as signal representations, signal manipulations, properties of signals along with the classification of a discrete-time system and its properties.

A. Discrete-time Signals

Discrete-time signals are represented mathematically by sequences of real or complex numbers. In particular, a discrete-time signal is a function of an integer-valued variable n, which is denoted by $x[n]$. The variable n often denotes time but can be used to represent other quantities. For example, a real-valued signal $x[n]$ can be illustrated by the following plot:

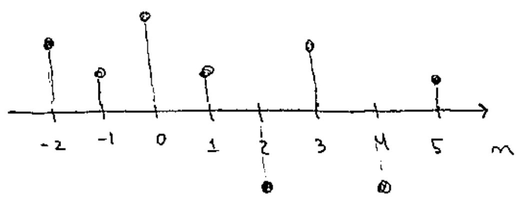


Fig. 1. A graphical representation of $x[n]$.

In some problems and applications, it is convenient to view $x[n]$ as a vector. Thus, the sequence values $x[0]$ to $x[N-1]$ can be considered as elements of a column vector as follows:

$$\underline{x}[n] = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Discrete-time signals are often derived by sampling a continuous-time signal, such as speech, audio and waveforms, with an analog-to-digital (A/D) converter. A continuous-time signal $x_a(t)$ that is sampled at a rate of $f_s = \frac{1}{T_s}$ samples per second produces the sampled signal $x[n]$, which is related to $x_a(t)$ by the following:

$$x[n] = x_a(n T_s),$$

where T_s is the sampling period and its reciprocal is the sampling frequency f_s.

However, it should be remarked that not all discrete-time signals are obtained by sampling. Some signals naturally occur as discrete-time signals because there is no physical analog-to-digital converter to transform $x_a(t)$ into $x[n]$. Examples of such signals include daily stock market prices, population statistics, currency values and inventories.

i) Complex sequences

In general, a discrete-time signal can be complex-values as often encountered in applications such as digital communications, radar and sonar systems. A complex signal signal can be expressed in terms of its real and imaginary parts:

$$z[n] = a[n] + j\, b[n] = \mathrm{Re}\{z[n]\} + j\, \mathrm{Im}\{z[n]\}$$

A complex signal can also be represented in polar form in terms of its magnitude and phase as given by

$$\begin{aligned} z[n] &= |z[n]| \exp\left(j \arg\{z[n]\}\right) \\ &= |z[n]| \exp(j\theta), \end{aligned}$$

where the magnitude is obtained from the real and the imaginary parts as follows:

$$|z[n]|^2 = \mathrm{Re}^2\{z[n]\} + \mathrm{Im}^2\{z[n]\},$$

whereas the phase Θ is given by

$$\Theta = \arg\{z[n]\} = \tan^{-1}\frac{\mathrm{Im}\{z[n]\}}{\mathrm{Re}\{z[n]\}}$$

If $z[n]$ is a complex sequence, the complex conjugate, denoted by $z^*[n]$, is formed by changing the sign on the imaginary part of $z[n]$:

$$z^*[n] = \mathrm{Re}\{z[n]\} - j\,\mathrm{Im}\{z[n]\}$$
$$= |z[n]| \exp\left(-j \arg\{z[n]\}\right)$$

ii) Some fundamental sequences

There are a few simple and yet important sequences that are widely-used in discrete-time signals and systems.

The unit sample is defined by

$$\delta[n] = \begin{cases} 1, & n=0 \\ 0, & n \neq 0 \end{cases}$$

and is illustrated by

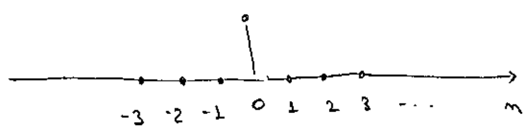
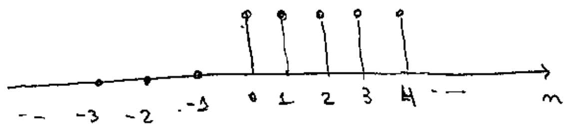


Fig. 2. Unit sample

The unit step is defined by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and is illustrated by



The unit step $u[n]$ is related to the unit sample $\delta[n]$ by

$$u[n] = \sum_{k=-\infty}^{n} \delta[k],$$

whereas a unit sample can be written as

$$\delta[n] = u[n] - u[n-1]$$

More generally, any sequence can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

Another important class of basic signals is that of exponential sequences, which have the general form given by

$$x[n] = a^n,$$

where a can be a real or complex number. Of particular interest is the exponential sequence that is formed when $a = e^{j w_0}$, where w_0 is called the frequency and is a real number. In this case, $x[n]$ is a complex exponential

$$e^{j n w_0} = \cos(n w_0) + j \sin(n w_0)$$

iii) Signal Duration

Discrete-time signals can be classified in terms of their duration. A discrete-time sequence is a finite-length sequence if it is equal to zero for all values of n outside a finite interval, otherwise the sequence is an infinite-length sequence

iv) Periodic and Aperiodic sequences

A discrete-time signal can be classified as periodic or aperiodic. A signal $x[n]$ is said to be periodic if

$$x[n] = x[n+N], \quad \text{for all } n$$

where N is a positive real integer. This means the sequence repeats itself every N samples. The fundamental period is N, which corresponds to the smallest positive integer for $x[n] = x[n+N]$ to hold. When the above equation is not satisfied, $x[n]$ is said to be aperiodic.

Ex 1: The signal $x_1[n] = e^{j\frac{\pi}{8}n}$ can be rewritten as

$$x_1[n] = e^{j\frac{2\pi}{16}n} = e^{j\frac{2\pi}{16}(n+N)}$$ where $N=16$ is the fundamental period which means it is periodic.

The signal $x_2[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

$x_2[n] = a^n \neq x_2[n+N] = a^{n+N}$. Thus, the sequence is aperiodic.

v) Symmetric sequences

A discrete-time signal often exhibits some form of symmetry that may be exploited in solving problems.

A real-valued signal is even if

$$x[n] = x[-n], \quad \text{for all } n,$$

whereas it is odd if

$$x[n] = -x[-n], \quad \text{for all } n$$

Any signal can be decomposed into a sum of its even part, $x_e[n]$ and its odd part, $x_o[n]$, as follows

$$x[n] = x_e[n] + x_o[n]$$

In order to find the even part we compute

$$x_e[n] = \frac{1}{2}\left(x[n] + x[-n]\right),$$

whereas to find the odd part we calculate

$$x_o[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$$

For complex signals, the symmetries are different. A complex signal is said to be conjugate symmetric if,

$$x[n] = x^*[-n], \quad \text{for all } n,$$

and a signal is said to be conjugate antisymmetric if

$$x[n] = -x^*[-n]$$

Any complex signal may always be decomposed into a sum of a conjugate symmetric signal and a conjugate antisymmetric signal.

vi) Signal Manipulations

In discrete-time signals and systems, we are often concerned with the manipulation of signals. These are based on signal transformations which can be classified as transformations of the independent variable n or of the amplitude of $x[n]$.

Sequences can be altered and manipulated by modifying the index n as follows:

$$y[n] = x[f[n]],$$

where $f[n]$ is some function of n.

The first transformation is called shifting and is defined by

$$f[n] = n - n_0,$$

where n_0 corresponds to a delay if n_0 is positive and to an advance if n_0 is negative. In particular, if $y[n] = x[n-n_0]$ is shifted to the right by n_0 samples if n_0 is positive, as illustrated in what follows,

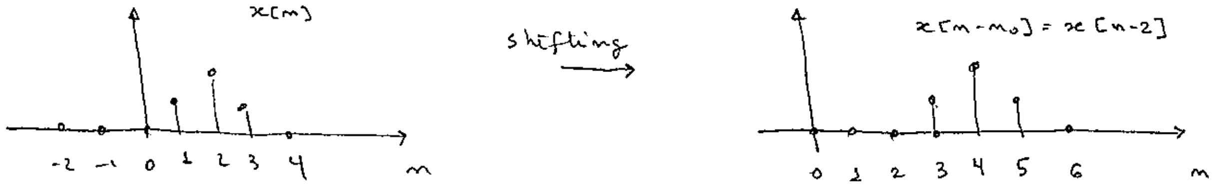


Fig 3. Shifting a signal by $n_0 = 2$ (delay)

Another transformation called reversal is given by

$$f[n] = -n,$$

where the effect is to flip a signal $x[n]$ with respect to the index n, as depicted below.

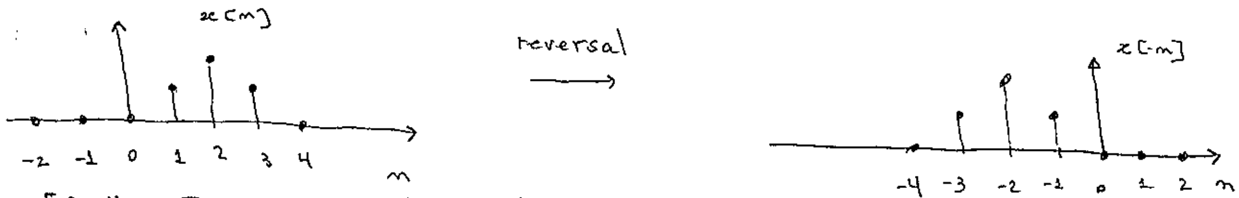


Fig. 4 Time reversal of a signal.

A time scaling transformation is defined by

$$f[n] = Mn \qquad \text{or} \qquad f[n] = \frac{n}{N},$$

where M and N are positive integers. In the case of $f[n] = M.n$, the sequence $x[Mn]$ is formed by taking every Mth sample of $x[n]$ which is known as down-sampling. With $f[n] = n/N$ the sequence $y[n] = x[f[n]]$ is defined by

$$y[n] = \begin{cases} x\left[\frac{n}{N}\right] & n = 0, \pm N, \pm 2N, \ldots \\ 0 & \text{otherwise} \end{cases}$$

which is known as up-sampling. These operations are illustrated below.

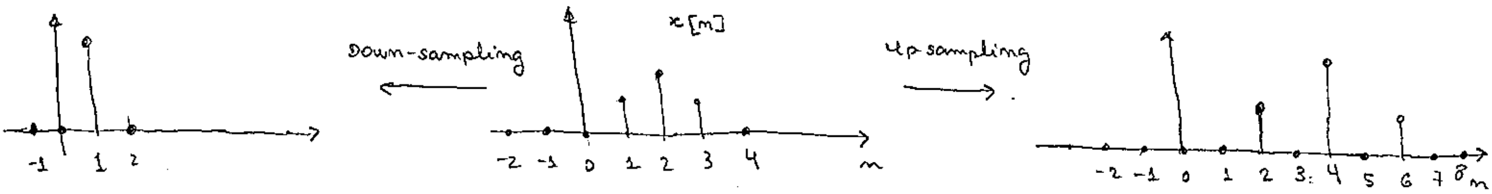


Fig. 5 Down- and up-sampling operations

Shifting, reversal and time-scaling operations are order-dependent which requires extra care in evaluating compositions of these operations. For example, let us consider the operations of delay and reversal in what follows

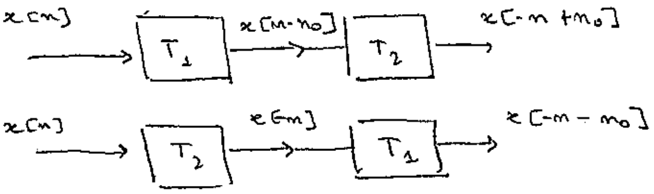


Fig. 6. order-dependent operations.

The most common types of amplitude transformations are addition, multiplication and scaling. Performing these operations is quite simple and only involves pointwise operations on the signal.

The addition operation involves the sum of two signals as given by

$$y[n] = x_1[n] + x_2[n], \qquad -\infty < n < \infty,$$

where $y[n]$ is formed by the pointwise product of signal values.

The multiplication of two signals is described by

$$y[n] = x_1[n] \cdot x_2[n], \qquad -\infty < n < \infty,$$

where $y[n]$ is formed by the pointwise product of the signal values.

Amplitude scaling of a signal $x[n]$ by a constant c is accomplished by multiplying every single value by c;

$$y[n] = c\, x[n], \qquad -\infty < n < \infty$$

vii) Signal Decomposition

The unit sample can be used to decompose an arbitrary signal $x[n]$ into a sum of weighted and shifted unit samples;

$$x[n] = \dots + x[-1] \cdot \delta[n+1] + x[0] \cdot \delta[n] + x[1] \cdot \delta[n-1] + x[2]\, \delta[n-2] + \dots$$

This decomposition can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot \delta[n-k],$$

where each term in the sum $x[k] \cdot \delta[n-k]$ has an amplitude $x[k]$ at time $n=k$ and a value of zero for all other values of n.

B. Discrete-Time Systems

A discrete-time system is a mathematical operator or mapping that transforms one signal (the input) into another signal (the output). by means of a fixed set of rules or operations. This mapping can be represented by

$$y[n] = T\big[\, x[n] \,\big],$$

where $x[n]$ and $y[n]$ are the input and output signals, respectively, and $T[\cdot]$ represents the system as depicted below.

$$x[n] \longrightarrow \boxed{T[\cdot]} \longrightarrow y[n]$$

Fig. 7 Representation of a discrete-time system.

Ex : Moving Average System

A general moving average system is defined by

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

This system computes the n^{th} sample of $y[n]$ as the average of $(M_1 + M_2 + 1)$ samples of the input sequence around the n^{th} sample. This is widely used for analyzing time series.

Discrete-time systems may be classified in terms of the properties they possess. The most important properties include linearity, shift-invariance, causality, stability and invertibility. and will be defined in what follows.

i) Memoryless Systems

A system is said to be memoryless if the output $y[n]$ at any time n depends only on the input $x[n]$ at time n.

Ex: A memoryless system

$y[n] = x^2[n]$ is memoryless because $y[n_0]$ depends only on $x[n_0]$

$y[n] = x[n] + x[n-1]$ is not memoryless because $y[n]$ depends on both $x[n]$ and $x[n-1]$.

ii) Linear Systems

If $y_1[n]$ and $y_2[n]$ are the responses of a system when $x_1[n]$ and $x_2[n]$ are the respective inputs, then the system is linear if and only if

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$

$$= y_1[n] + y_2[n]$$

and

$$T\{a x[n]\} = a T\{x[n]\} = a\, y[n],$$

where a is an arbitrary constant. The first property is the additive property and the second the homogeneity or scaling property. These two properties together comprise the principle of superposition stated as

$$T\{a x_1[n] + b x_2[n]\} = a T\{x_1[n]\} + b T\{x_2[n]\},$$

for arbitrary constants a and b.

Ex: The system defined by

$$y[n] = \frac{x^2[n]}{x[n-1]}$$

is not additive because

$$T[x_1[n] + x_2[n]] = \frac{(x_1[n] + x_2[n])^2}{x_1[n-1] + x_2[n-1]}$$

which is not the same as

$$T[x_1[n]] + T[x_2[n]] = \frac{x_1^2[n]}{x_1[n-1]} + \frac{x_2^2[n]}{x_2[n-1]}$$

but it is homogeneous because for an input $c x[n]$ the output is

$$T[c\, x[n]] = \frac{(c\, x[n])^2}{c\, x[n-1]} = c \cdot \frac{x^2[n]}{x[n-1]} = c\, T[x[n]]$$

iii) Time-Invariant Systems.

A time-invariant system is a system for which a time shift or delay of the input sequence causes a corresponding shift in the output sequence. Let $y[n]$ be the response of a system to an arbitrary input $x[n]$. The system is said to be shift-invariant if, for any delay n_0, the response to $x[n-n_0]$ is $y[n-n_0]$. A system that is not shift-invariant is said to be shift-varying.

Ex: The system defined by

$$y[n] = x^2[n]$$

is shift-invariant, which may be shown as follows. If $y[n] = x^2[n]$ is the response of the system to $x[n]$, the response of the system to $x'[n] = x[n-n_0]$ is

$$y'[n] = (x'[n])^2 = x^2[n-n_0]$$

Because $y'[n] = y[n-n_0]$, the system is time-invariant.

iv) Causality

A system is causal if for any n_0 the response of the system at time n_0 depends only on the input up to time $n=n_0$. For a causal system, changes in the output cannot precede changes in the input. Thus if $x_1[n] = x_2[n]$ for $n \leq n_0$, $y_1[n]$ must be equal to $y_2[n]$ for $n \leq n_0$.

Ex: The system described by

$$y[n] = x[n] + x[n-1]$$

is causal because the value of the output at time $n=n_0$ depends only on the input $x[n]$ at time n_0 and at time n_0-1.

The system described by

$$y[n] = x[n] + x[n+1]$$

is non causal because the output at time $n=n_0$ depends on the value of the input at time n_0+1.

v) Stability

In many applications, it is important for a system to have a response, $y[n]$, that is bounded in amplitude whenever the input is bounded. A system with this property is said to be stable in the bounded input-bounded output (BIBO) sense. Specifically, a system is said to be stable in the BIBO sense if, for any input that is bounded, $|x[n]| \leq A < \infty$, the output will be bounded,

$$|y[n]| \leq B < \infty$$

For a linear time-invariant system, stability is guaranteed if the unit sample response is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Ex: A linear time-invariant system with unit sample response given by

$$h[n] = a^n u[n]$$

will be stable whenever $|a| < 1$, because

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} \qquad |a| < 1$$

On the other hand, the system described by

$$y[n] = n\,x[n]$$

is not stable because the response to a unit step, $x[n] = u[n]$ is $y[n] = n\,u[n]$, which is unbounded.

c. Linear Time-Invariant Systems

An important class of discrete-time systems consists of those that are both linear and time-invariant. Such systems can be completely characterised by their impulse responses.

Let $h_k[n]$ be the response of the system to the input $\delta[n-k]$ at time $n=k$. Then, the output of the system is given by

$$y[n] = T\left\{ \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k] \right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k].T\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k].\, h_k[n].$$

Using the property of time invariant system, we have

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].\, h[n-k], \quad \text{for all } n$$

which shows that the above equation can be used to compute each sample of $y[n]$ given $h[n]$ and $x[n]$. This equation is also referred to as a convolution sum and can be represented by

$$y[n] = h[n] * x[n],$$

where $*$ is the convolution operator.

Ex: Let us perform the convolution of the signals

$$x[n] = a^n u[n] = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

and

$$h[n] = u[n]$$

The convolution sum is given by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k].\, h[n-k] = \sum_{k=-\infty}^{\infty} a^k u[k].u[n-k]$$

Because $u[k]$ is equal to zero for $k<0$ and $u[n-k]$ is equal to zero for $k>n$ when $n<0$, there are nonzero terms in the sum and $y[n]=0$. If $n \geq 0$, we have

$$y[n] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

Therefore, we have

$$y[n] = \frac{1-a^{n+1}}{1-a}.u[n]$$

Since all linear time-invariant systems are described by a convolution sum, the properties of this class of systems are defined by the properties of the convolution sum. Moreover, the impulse response $h[n]$ is a complete characterisation of the properties of a linear time-invariant system.

i) Commutative property :

The convolution operation is commutative as described by

$$x[n] * h[n] = h[n] * x[n]$$

From a systems point of view, this property states that a system with an impulse response $h[n]$ and input $x[n]$ behaves in exactly the same way as a system with impulse response $x[n]$ and input $h[n]$, as illustrated below

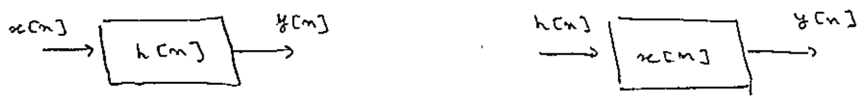


Fig. 8 The commutative property of LTI systems

ii) Associative Property :

The convolution operation satisfies the associative property described by

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\}$$

From a systems point of view, this property states that if two systems with $h_1[n]$ and $h_2[n]$ are connected in cascade, an equivalent system is one that has an impulse response given by the convolution of $h_1[n]$ and $h_2[n]$, as depicted below.

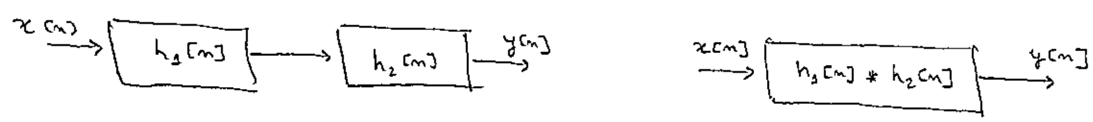


Fig. 9. The associative property of LTI systems.

iii) Distributive Property :

The distributive property of the convolution operator states that

$$x[n] * \{ h_1[n] + h_2[n] \} = x[n] * h_1[n] + x[n] * h_2[n]$$

From a systems perspective, this property asserts that if two systems with impulse responses $h_1[n]$ and $h_2[n]$ are connected in parallel, an equivalent system corresponds to the sum of $h_1[n]$ and $h_2[n]$, as illustrated below:

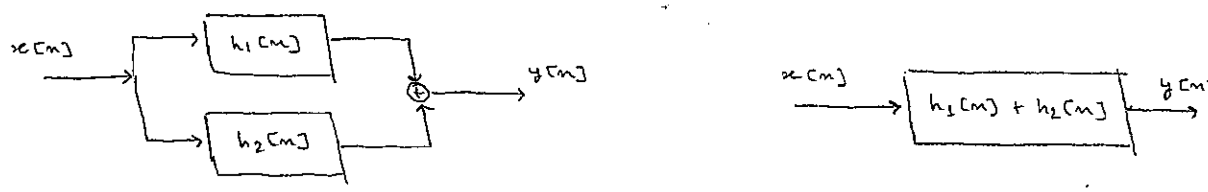


Fig. 10 The distributive property of LTI systems.

Ex: Consider the cascade connection of

$$h_1[n] = \sum_{k=-\infty}^{n} \delta[k] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} = u[n]$$

and

$$h_2[n] = \delta[n] - \delta[n-1]$$

and compute the impulse response of the cascade system.

The impulse response $h_1[n]$ corresponds to an accumulator and that of $h_2[n]$ is known as the backward difference system. The impulse response of the cascade connection is given by

$$\begin{aligned} h[n] &= h_1[n] * h_2[n] \\ &= u[n] * (\delta[n] - \delta[n-1]) \\ &= u[n] - u[n-1] \\ &= \delta[n] \end{aligned}$$

which yields a system whose overall impulse response is the impulse and introduces the concept of an inverse system. We say that $h_2[n]$ is the inverse system of $h_1[n]$ and vice-versa.

D. Difference Equations

An important class of LTI discrete-time systems is that characterised by linear constant coefficient difference equations. They provide a more efficient way to represent discrete-time LTI systems and have the general form given by

$$y[n] = \sum_{k=0}^{q} b[k] . x[n-k] - \sum_{k=1}^{p} a[k] \, y[n-k],$$

where the coefficients $a[k]$ and $b[k]$ are constants that define the system. If all the coefficients $a[k]$ are zero the difference equation is nonrecursive, otherwise if $a[k]$ are nonzero the difference equation is recursive.

Ex: A system with impulse response given by $h[n] = \alpha^n u[n]$ is described by

$$y[n] = \sum_{k=0}^{\infty} \alpha^k x[n-k]$$

This system can be represented by

$$y[n] = \alpha \, y[n-1] + x[n]$$

which is an example of a first-order recursive difference equation.

Difference equations provide a method for computing the response of a system given by $y[n]$ to an arbitrary input $x[n]$. Prior to their solution, it is necessary to specify a set of initial conditions. When these initial conditions are zero, the system is in initial rest.

For an LTI system described by a difference equation, $h[n]$ is found by solving the difference equation for $x[n] = \delta[n]$ assuming initial rest. For a nonrecursive system $(a[k] = 0)$, the difference equation becomes

$$y[n] = \sum_{k=0}^{q} b[k] \, x[n-k]$$

and the output is simply a weighted sum of the current and past values

As a result, the impulse response $h[n]$ is given by

$$h[n] = \sum_{k=0}^{q} b[k]\,\delta[n-k]$$

Thus, $h[n]$ is finite in length and the system is referred to as a finite-length impulse response (FIR) system. However, if $a[k] \neq 0$, the impulse response is, in general, infinite in length and the system is referred to as an infinite length impulse response (IIR) system.

There are several methods to solve difference equations for an arbitrary input $x[n]$. The classical approach is to find the homogeneous and particular solutions which is detailed in what follows.

The general solution of a difference equation is a sum of two parts:

$$y[n] = y_h[n] + y_p[n],$$

where $y_h[n]$ is known as the homogeneous solution and $y_p[n]$ is the particular solution. The homogeneous solution is the response of the system to the initial conditions and $x[n] = 0$. The particular solution is the response of the system to $x[n]$ with zero initial conditions.

The homogeneous solution is found by solving

$$y[n] + \sum_{k=1}^{p} a[k] \cdot y[n-k] = 0,$$

which relies on a solution of the form

$$y_h[n] = z^n$$

Substituting the above into the previous equation leads to

$$z^n + \sum_{k=1}^{p} a[k]\, z^{n-k} = 0$$

or

$$z^{n-p}\left\{ z^p + a[1]\,z^{p-1} + \dots + a[p-1]\,z + a[p] \right\} = 0$$

The polynomial in braces is called the characteristic polynomial. Because it has degree p, it will have p roots which may be either real or complex. If the coefficients $a[k]$ are real-valued, these roots will occur in complex conjugate pairs. If the p roots z_i are distinct, $z_i \neq z_k$ for $k \neq i$, the general solution to the homogeneous difference equation is

$$y_h[n] = \sum_{k=1}^{p} A_k z_k^n ,$$

where the constants are chosen to satisfy the initial conditions.

For repeated roots, the solution must be modified as follows. If z_1 is a root of multiplicity m with the remaining $p-m$ roots distinct, the homogeneous solution becomes

$$y_h[n] = (A_1 + A_2 n + \dots + A_m n^{m-1}) z_1^n + \sum_{k=m+1}^{p} A_k z_k^n$$

For the particular solution, it is necessary to find the sequence $y_p[n]$ that satisfies the difference equation for $x[n]$. In general, this requires some creativity and insight. However, for many of the typical inputs that we are interested in the solution will have the same form as the input. For example, if $x[n] = a^n u[n]$, the particular solution will be of the form

$$y_p[n] = C a^n u[n] ,$$

provided a is not a root of the characteristic equation. The constant C is found by substituting the solution into the difference equation. Note that for $x[n] = C\delta[n]$ the particular solution is zero. The table below list the particular solution for some common inputs.

Table I:

Term in $x[n]$	Particular Solution
C	C_1
Cn	$C_1 + C_2$
Ca^n	$C_1 a^n$
$C\cos(n\omega_0)$	$C_1 \cos(n\omega_0) + C_2 \sin(n\omega_0)$

Ex : Let us find the solution to

$$y[n] - 0.25\, y[n-2] = x[n]$$

for $x[n] = u[n]$ assuming initial conditions $y[-1] = 1$ and $y[-2] = 0$.

The particular solution is

$$y_p[n] = C_1$$

Substituting the above into the difference equation we find

$$C_1 - 0.25\, C_1 = 1$$

In order for this to hold, we must have $C_1 = 4/3$

We set $y_h[n] = z^n$ to find the homogeneous solution, which gives

$$z^2 - 0.25 = 0 \quad \text{or} \quad (z+0.5)(z-0.5) = 0$$

Therefore, the homogeneous solution has the form

$$y_h[n] = A_1\,(0.5)^n + A_2\,(-0.5)^n$$

The total solution is

$$y[n] = \frac{4}{3} + A_1(0.5)^n + A_2(-0.5)^n \quad , \; n \geq 0$$

The constants A_1 and A_2 can be found using the initial conditions:

$$y[0] - 0.25\, y[-2] = x[0] = 1$$
$$y[1] - 0.25\, y[-1] = x[1] = 1$$

which yields $A_1 = -\frac{1}{2}$ and $A_2 = \frac{1}{6}$

The solution is given by

$$y[n] = \frac{4}{3} - \frac{1}{2}(0.5)^n + \frac{1}{6}(-0.5)^n \quad , \; n \geq 0$$

E. Frequency-Domain Representation of Discrete-Time Signals and Systems

In this section, we will look at how complex exponentials are eigenfunctions of LTI systems and how this property leads to the notion of a frequency representation of LTI systems.

Eigenfunctions of LTI systems are sequences that when applied to a system pass through with only a change in (complex) amplitude. If the input is $x[n]$ then the output is $y[n] = \lambda x[n]$, where λ the eigenvalue depends only on $x[n]$, as illustrated below

$$\xrightarrow{x[n]} \boxed{h[n]} \longrightarrow y[n] = \lambda x[n]$$

Fig. 11 An eigenfunction of an LTI system.

Signals of the form $x[n] = e^{jn\omega}$, $-\infty < n < \infty$, where ω is a constant are eigenfunctions of LSI systems. This can be shown using the convolution sum:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, e^{j\omega(n-k)} = e^{jn\omega} \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-jk\omega}$$

$$= H(e^{j\omega})\, e^{jn\omega}$$

Thus, the eigenvalue, which we denote by $H(e^{j\omega})$, is

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{-jk\omega}$$

Note that $H(e^{j\omega})$ is, in general, complex-valued and depends on the frequency ω of the complex exponential. Thus, it may be rewritten in terms of its real and imaginary parts,

$$H(e^{j\omega}) = H_R(e^{j\omega}) + j\, H_I(e^{j\omega})$$

or in terms of its magnitude and phase,

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{j\phi_h(\omega)}$$

where $|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})$ and $\phi_h(\omega) = \tan^{-1} \dfrac{H_I(e^{j\omega})}{H_R(e^{j\omega})}$

Graphical representations of the frequency response are of great value in the analysis of LTI systems and plots of the magnitude and phase are commonly used. Another useful representation is a plot of $20 \log |H(e^{j\omega})|$ versus ω, as illustrated below.

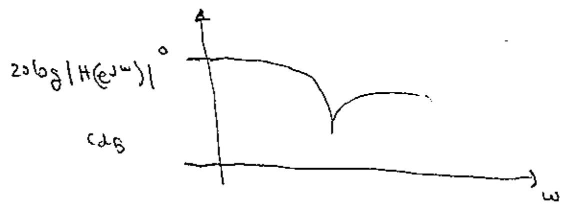


Fig. 12 frequency response of an LTI system.

Another representation of LTI systems is the group delay defined by

$$\tau_h(\omega) = -\frac{d\phi_h(\omega)}{d\omega}$$

The function $H(e^{j\omega})$ is known as frequency response and is very useful in the characterisation of LTI systems. For example, the response of an LTI system to the input

$$x[n] = \sum_{k=1}^{N} \alpha_k \cdot e^{jn\omega_k}$$

is given by

$$y[n] = \sum_{k=1}^{N} \alpha_k\, H(e^{j\omega_k})\, e^{jn\omega_k},$$

where $H(e^{j\omega_k})$ is the frequency response of the system at frequency ω_k.

Ex: Let $x[n] = \cos(n\omega_0) = \frac{1}{2} e^{jn\omega_0} + \frac{1}{2} e^{-jn\omega_0}$ be the input to an LTI system with a real-valued impulse response $h[n]$.

The response of the system is $y[n] = \frac{1}{2} H(e^{j\omega_0})\, e^{jn\omega_0} + \frac{1}{2} H(e^{-j\omega_0})\, e^{-jn\omega_0}$

Since $h[n]$ is real-valued, $H(e^{j\omega})$ is conjugate symmetric, we have $H(e^{-j\omega}) = H^*(e^{j\omega})$

Therefore, $y[n] = \frac{1}{2} H(e^{j\omega_0})\, e^{j\omega n_0} + \frac{1}{2} H^*(e^{j\omega_0})\, e^{-jn\omega_0}$ and it follows that

$$y[n] = \mathrm{Re}\{H(e^{j\omega_0})\, e^{j\omega_0}\} = |H(e^{j\omega_0})| \cos(n\omega_0 + \phi_h(\omega_0))$$

i) Periodicity

The frequency response is a complex-valued function of ω and is periodic with a period 2π. This is because a discrete-time complex exponential of frequency ω_0 has the relation

$$x[n] = e^{jn\omega_0} = e^{jn(\omega_0 + 2\pi)}$$

Therefore, if the input to an LTI system is $x[n] = e^{jn\omega_0}$ the response must be the same as that to the signal $x[n] = e^{jn(\omega_0 + 2\pi)}$, which requires that

$$H(e^{j\omega_0}) = H(e^{j(\omega_0 + 2\pi)})$$

ii) Symmetry

If $h[n]$ is real-valued, the frequency response is a conjugate symmetric function of frequency:

$$H(e^{-j\omega}) = H^*(e^{j\omega}).$$

Conjugate symmetry of $H(e^{j\omega})$ implies that the real part is an even function of ω

$$H_R(e^{j\omega}) = H_R(e^{-j\omega})$$

and that the imaginary part is odd

$$H_I(e^{j\omega}) = -H_I(e^{-j\omega})$$

Conjugate symmetry also implies that the magnitude is even

$$|H(e^{j\omega})| = |H(e^{-j\omega})|$$

and that the phase and group delay are odd

$$\phi_h(\omega) = -\phi_h(-\omega) \quad \text{and} \quad \tau_h(\omega) = -\tau_h(-\omega)$$

Ex: Consider the LTI system given by $h[n] = \alpha^n u[n]$, where α is a real number and $|\alpha| < 1$.

The frequency response is described by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \cdot e^{-jn\omega} = \sum_{n=0}^{\infty} \alpha^n e^{-jn\omega}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}}$$

iii) Inverting the Frequency Response

The frequency response of an LTI system is given by

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-jnw} .$$

The inverse frequency response of an LTI system retrieves the impulse response as described by

$$h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw})\, e^{jw}\, dw$$

Ex: For a system with a frequency response given by

$$H(e^{jw}) = \begin{cases} 1 & |w| \leq w_c \\ 0 & w_c < |w| \leq \pi \end{cases}$$

This corresponds to an ideal low-pass filter. The impulse response is obtained by

$$h[n] = \frac{1}{2\pi} \int_{-w_c}^{w_c} e^{jnw}\, dw = \frac{1}{2j\pi n}\left[e^{jnw_c} - e^{-jnw_c} \right] = \frac{\sin n w_c}{\pi n}$$

This system is noncausal, unstable and unrealizable.

F. The Discrete-Time Fourier Transform

The discrete-time Fourier transform (DTFT) of a sequence $x[n]$ is defined by

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-jnw}$$

In order for the DTFT of a sequence to exist, the summation above must converge, which requires that $x[n]$ be absolutely summable:

$$\sum_{n=-\infty}^{\infty} |x[n]| = S < \infty$$

Ex; The DTFT of the sequence $x_1[n] = \alpha^n u[n]$, $|\alpha| < 1$ is

$$X_1(e^{jw}) = \sum_{n=0}^{\infty} \alpha^n e^{-jnw} = \sum_{n=0}^{\infty} (\alpha e^{-jw})^n$$

Using the geometric series, this sum yields

$$X_1(e^{jw}) = \frac{1}{1 - \alpha e^{-jw}} \quad \text{provided } |\alpha| < 1.$$

Similarly, for the sequence $x_2[n] = -\alpha^n u[-n-1]$, $|\alpha| > 1$

the DTFT is

$$X_2(e^{jw}) = \sum_{n=-\infty}^{\infty} x_2[n] \cdot e^{-jnw} = -\sum_{n=-\infty}^{-1} \alpha^n e^{-jnw}$$

Changing the limits on the sum, we have

$$X_2(e^{jw}) = -\sum_{n=1}^{\infty} \alpha^{-n} e^{jnw} = -\sum_{n=0}^{\infty} (\alpha^{-1} e^{jw})^n + 1$$

If $|\alpha| > 1$, this sum is

$$X_2(e^{jw}) = -\frac{1}{1 - \alpha^{-1} e^{jw}} + 1 = \frac{1}{1 - \alpha e^{-jw}}$$

Therefore, $x_1[n] = \alpha^n u[n]$ and $x_2[n] = -\alpha^n u[-n-1]$ have the same DTFT.

The inverse DTFT is given by

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{jw}) e^{jnw} \, dw$$

which corresponds to a decomposition of $x[n]$ into a linear combination of all complex exponentials with frequencies in the range $-\pi < w < \pi$.

Some useful DTFT pairs are included in the Table below

Table 2 : Some common DTFT pairs

Sequence	DTFT
$\delta[n]$	1
$\delta[n-n_o]$	$e^{-jn_o w}$
1	$2\pi\,\delta[w]$
e^{jnw_o}	$2\pi\,\delta[w-w_o]$
$a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{1 - a e^{-jw}}$
$(n+1) a^n u[n]$, $\lvert a\rvert < 1$	$\frac{1}{(1 - a e^{-jw})^2}$
$\cos n w_o$	$\pi\,\delta[w + w_o] + \pi\,\delta[w - w_o]$

Ex: suppose that $X(e^{jw})$ consists of an impulse at frequency $w = w_0$

$$X(e^{jw}) = \delta[w - w_0]$$

Using the inverse DTFT, we have

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jnw}\, dw = \frac{1}{2\pi} \cdot e^{jnw_0}$$

Ex: If $X(e^{jw}) = \pi\,\delta[w - w_0] + \pi\,\delta[w + w_0]$

the the inverse DTFT is given by

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jnw}\, dw = \frac{1}{2} e^{jnw_0} + \frac{1}{2} e^{-jnw_0} = \cos(nw_0)$$

The DTFT has a number of properties that are useful to simplify its computation and of its inverse. In what follows, we review them.

i) Periodicity

The DTFT is periodic in w with a period of 2π:

$$X(e^{jw}) = X(e^{j(w+2\pi)})$$

ii) Symmetry

The DTFT has the following symmetries:

$x[n]$	$X(e^{jw})$
Real and even	Real and even
Real and odd	Imaginary and odd
Imaginary and even	Imaginary and even
Imaginery and odd	Real and odd

iii) Linearity

The DTFT is a linear operator as described by

$$a\,x_1[n] + b\,x_2[n] \xleftrightarrow{DTFT} a\,X_1(e^{jw}) + b\,X_2(e^{jw})$$

iv) Shifting property

When a sequence is shifted in time, its DTFT is multiplied by a complex exponential as described by

$$x[n - n_0] \xleftrightarrow{DTFT} e^{-jn_0 w} X(e^{jw})$$

v) Time-Reversal

Time-reversing a sequence results in a frequency reversal of the DTFT as described by:

$$x[-n] \xleftrightarrow{DTFT} X(e^{-j\omega})$$

vi) Modulation

Multiplying a sequence by a complex exponential results in a frequency shift of the DTFT:

$$e^{jn\omega_0} x[n] \xleftrightarrow{DTFT} X(e^{j(\omega-\omega_0)})$$

Thus, modulating a sequence by a cosine of frequency ω_0 shifts the spectrum up and down in frequency by ω_0,

$$x[n] \cos n\omega_0 \xleftrightarrow{DTFT} \frac{1}{2} X(e^{j(\omega-\omega_0)}) + \frac{1}{2} X(e^{j(\omega+\omega_0)})$$

vii) Convolution Theorem

One of the most important results in linear systems theory is that convolution in the time domain is equivalent to multiplication in the frequency domain. This theorem states that the DTFT of a sequence that is formed by convolving two sequences, $x[n]$ and $h[n]$, is the product of the DTFTs of $x[n]$ and $h[n]$:

$$h[n] * x[n] \xleftrightarrow{DTFT} H(e^{j\omega}) \cdot X(e^{j\omega})$$

viii) Multiplication (Periodic Convolution) Theorem

There is a dual to the convolution theorem that states that multiplication in the time domain corresponds to (periodic) convolution in the frequency domain:

$$x[n] \cdot y[n] \xleftrightarrow{DTFT} \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) \, d\theta$$

ix) Parseval's Theorem

A corollary to the multiplication Theorem is Parseval's theorem:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

Parseval's theorem is an energy conservation theorem because it states that the DTFT operator preserves energy when going from the time domain to the frequency domain.

G Applications

The applications of the DTFT include finding the frequency response of LTI systems described by difference equations, performing convolutions and designing inverse systems.

i) LTI systems and difference equations

Consider the difference equation given by

$$y[n] = -\sum_{k=1}^{p} a[k]\, y[n-k] + \sum_{k=0}^{q} b[k]\, x[n-k]$$

The linearity and shift properties of the DTFT may be used to express this difference equation in the frequency domain as follows:

$$Y(e^{jw}) = -\sum_{k=1}^{p} a[k]\, e^{-jkw}\, Y(e^{jw}) + \sum_{k=0}^{q} b[k]\, e^{-jkw}\, X(e^{jw})$$

or

$$Y(e^{jw}) \left[1 + \sum_{k=1}^{p} a[k]\, e^{-jkw} \right] = X(e^{jw}) \sum_{k=0}^{q} b[k]\, e^{-jkw}$$

Therefore, the frequency response of this system is

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{k=0}^{q} b[k] \cdot e^{-jkw}}{1 + \sum_{k=1}^{p} a[k]\, e^{-jkw}}$$

Ex: Consider the LTI system characterised by

$$y[n] = 1.34\, y[n-1] - 0.90\, y[n-2] + x[n] - 1.41\, x[n-1] + x[n-2]$$

The frequency response may be found by inspection

$$H(e^{jw}) = \frac{1 - 1.41\, e^{-jw} + e^{-2jw}}{1 - 1.34\, e^{-jw} + 0.90\, e^{-2jw}}$$

Ex: Consider the frequency response given by

$$H(e^{j\omega}) = \frac{1 + e^{-2j\omega}}{2 - e^{-j\omega} + 0.5\, e^{-j2\omega}} = \frac{0.5 + 0.5\, e^{-2j\omega}}{1 - 0.5\, e^{-j\omega} + 0.25\, e^{-2j\omega}}$$

A difference equation for this system is

$$y[n] = 0.5\, y[n-1] - 0.25\, y[n-2] + 0.5\, x[n] + 0.5\, x[n-2]$$

ii) Computing Convolutions

Since the DTFT maps convolution in the time domain into multiplication in the frequency domain, it provides a way to compute convolutions in the time domain

Ex: Consider the impulse response of an LTI system given by

$$h[n] = \alpha^n . u[n]$$

The response of the system to the input $x[n] = \beta^n u[n]$, where $|\alpha| < 1$, $|\beta| < 1$ and $\alpha \neq \beta$, is given by

$$y[n] = h[n] * x[n]$$

The DTFT of $y[n]$ is $Y(e^{j\omega}) = H(e^{j\omega}) . X(e^{j\omega}) = \dfrac{1}{1-\alpha e^{-j\omega}} . \dfrac{1}{1-\beta e^{-j\omega}}$

By expanding $Y(e^{j\omega})$ we have

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})} = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}} = \frac{\alpha / \alpha-\beta}{1-\alpha e^{-j\omega}} - \frac{\beta / \alpha-\beta}{1-\beta e^{-j\omega}}$$

The inverse DTFT is given by

$$y[n] = \left[\frac{\alpha}{\alpha-\beta}\, \alpha^n - \frac{\beta}{\alpha-\beta}\, \beta^n \right] u[n]$$

iii) Solving Difference Equations

The DTFT may be used to solve difference equations in the frequency domain provided that the initial conditions are zero. The procedure relies on finding the DTFT of each term, solving for each term and then finding the inverse DTFT.

Ex : Consider the difference equation for $y[n]$ with zero initial conditions

$$y[n] - 0.25\, y[n-1] = x[n] - x[n-2]$$

for $x[n] = \delta[n]$. By taking the DTFT of each term, we have

$$Y(e^{j\omega}) - 0.25\, e^{-j\omega} Y(e^{j\omega}) = X(e^{j\omega}) - e^{-2j\omega} X(e^{j\omega})$$

Because the DTFT of $x[n]$ is $X(e^{j\omega}) = 1$, we have

$$Y(e^{j\omega}) = \frac{1 - e^{-j2\omega}}{1 - 0.25\, e^{-j\omega}} = \frac{1}{1 - 0.25\, e^{-j\omega}} - \frac{e^{-j2\omega}}{1 - 0.25\, e^{-j\omega}}$$

Using the DTFT pair $(0.25)^n u[n] \longleftrightarrow \dfrac{1}{1 - 0.25\, e^{-j\omega}}$

The inverse DTFT of $Y(e^{j\omega})$ is given by

$$y[n] = (0.25)^n u[n] - (0.25)^{n-2} u[n-2]$$

iv) Inverse Systems

The inverse of a system with impulse response $h[n]$ is a system that has impulse response $g[n]$ such that

$$h[n] * g[n] = \delta[n]$$

In terms of frequency response, the inverse of $H(e^{j\omega})$ (if it exists) is given by

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

Ex : If the frequency response of an LTI system is

$$H(e^{j\omega}) = \frac{1 - \frac{1}{4} e^{-j\omega}}{1 + \frac{1}{2} e^{-j\omega}}$$

then the inverse system is $G(e^{j\omega}) = \dfrac{1}{H(e^{j\omega})} = \dfrac{1 + \frac{1}{2} e^{-j\omega}}{1 - \frac{1}{4} e^{-j\omega}}$

which has an impulse response given by

$$g[n] = (0.25)^n u[n] + 0.5\, (0.25)^{n-1} u[n-1]$$