# Sensor Array Signal Processing

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# Syllabus

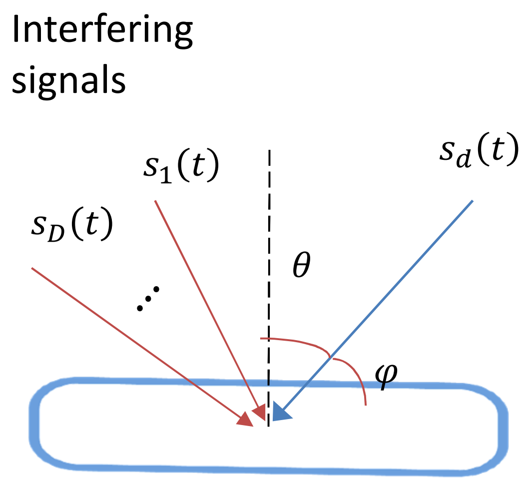
I. Introduction
- Fundamentals
- Sensor arrays
- Discrete-time models

II. Beamforming
- Main principles
- Optimum beamforming
- Robust beamforming
- Adaptive algorithms

III. Direction finding
- Maximum likelihood estimation
- Cramèr-Rao lower bound
- Capon´s technique
- MUSIC
- ESPRIT

# III. Direction finding

- Aims:
  - To compute the direction (or angle) of arrival of a signal impinging on a sensor array.
  - The angle $\theta$ is required to construct the steering vector used for beamforming.




- Direction finding principles:
  - Computation of statistical information of signals received by an array of sensors.
  - A method that employs statistical information and knowledge about the sensor array is employed to determine the angle of arrival.

# Signal model

- Let us consider a ULA with N sensors and the following signal model:

$$\boldsymbol{x}[i] = \mathbf{a}(\theta)s[i] + \boldsymbol{j}[i] + \boldsymbol{n}[i] \in \mathbb{C}^N$$

$$= \sum_{d=1}^{D} s_d[i]\mathbf{a}(\theta_d) + \boldsymbol{n}[i]\,,$$

$$= \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}[i] + \boldsymbol{n}[i],$$

where $\mathbf{a}(\theta_d) \in \mathbb{C}^N$ is the steering vector of the dth signal

$\boldsymbol{A}(\boldsymbol{\theta}) = [\mathbf{a}(\theta_1)|\cdots|\mathbf{a}(\theta_D)] \in \mathbb{C}^{N\times D}$ is the array manifold

$\boldsymbol{s}[i] \in \mathbb{C}^N$ is the vector with the signals impinging on the array

$\boldsymbol{n}[i] \in \mathbb{C}^N$ is the noise vector

$\boldsymbol{s}[i]$ and $\boldsymbol{n}[i]$ are statistically independent

$\mathbf{a}(\theta_d)$ are linearly independent

# A. Maximum likelihood estimation

- Consider the signal model of a ULA with N sensors given by

$$\boldsymbol{x}[i] = \mathbf{a}(\theta_d)s[i] + \boldsymbol{j}[i] + \boldsymbol{n}[i] \in \mathbb{C}^N$$

$$= \sum_{\boldsymbol{d}=\mathbf{1}}^{\boldsymbol{D}} s_d[i]\mathbf{a}(\theta_d) + \boldsymbol{n}[i] \, ,$$

where $\boldsymbol{j}[i]$ has known statistical information and the steering vectors are linearly independent.

- The noise samples are independent and identically distributed and taken from a complex stochastic process of zero mean and variance ${\sigma_n}^2$.

- Maximum likelihood (ML) estimation considers the joint probability density function of $\boldsymbol{x}[i]$ given $\theta_d$ as described by

$$p_{X|\Theta_d}(\boldsymbol{x}[i]|\ \theta_d) = \frac{1}{\Pi^N \det(\boldsymbol{R})} e^{-\boldsymbol{x}[i]^H \boldsymbol{R}^{-1} \boldsymbol{x}[i]}$$

# Maximum likelihood estimator

- The ML estimator can be computed as follows:

$$\widehat{\Theta}_d = \arg\max_{\Theta_d} p_{X|\Theta_d}(\boldsymbol{x}[i]|\ \theta_d)$$

- Since the natural logarithm is a monotonically increasing function, it is equivalent to compute

$$\begin{aligned}\widehat{\Theta}_d &= \arg\max_{\Theta_d} \ln p_{X|\Theta_d}(\boldsymbol{x}[i]|\ \theta_d) \\ &= \arg\max_{\Theta_d} -\left[\ln\det(\boldsymbol{R}) + \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}^H[l]\boldsymbol{R}^{-1}\boldsymbol{x}[l]\right] \\ &= \arg\min_{\Theta_d} \left[\ln\det(\boldsymbol{R}) + \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}^H[l]\boldsymbol{R}^{-1}\boldsymbol{x}[l]\right]\end{aligned}$$

# Important notes about ML estimation

- The function $P_{ML}(\theta_d) = \left[\ln \det(\boldsymbol{R}) + \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}^H[l]\boldsymbol{R}^{-1}\boldsymbol{x}[l]\right]$ is the ML estimator for the sensor data $\boldsymbol{x}[i]$.

- The ML estimate is the angle $\hat{\theta}_d$ associated to the maximum of $P_{ML}(\theta_d)$.

- The ML estimate requires a search that finds the peaks of $P_{ML}(\theta_d)$ in a dense grid with small angular spacing (e.g. 0.1 degree)

- The ML estimate is asymptotically efficient, i.e., approaches the CRLB.

# Performance analysis

- An often used quality measure for the parameter estimates, including ML estimates, is the Cramèr-Rao lower bound (CRLB).

- From estimation theory, we know that the variance of the unbiased estimate of $\theta_d$ is related to the CRLB by

$$var(\theta_d) \geq \boldsymbol{J}_{dd}{}^{-1},$$

where $J_{dd}{}^{-1}$ is the diagonal entry of the inverse of the Fisher information matrix $J$ whose ij-th element is given by

$$\boldsymbol{J}_{ij} = -E\left[\frac{\partial^2}{\partial\theta_i\partial\theta_j}\ln p_{X|\Theta_d}(\boldsymbol{x}[i]|\ \theta_d)\right]$$

# Performance analysis

- If we consider the signal model of a ULA and compute the partial derivatives in the expression of CRLB, we arrive at

$$var(\theta_d) \geq \frac{6{\sigma_n}^2}{{\sigma_s}^2 N(N^2-1)}$$
$$= \frac{6}{SNR\ N(N^2-1)},$$

where $SNR = \frac{{\sigma_s}^2}{{\sigma_n}^2}$ contributes to the reduction of the CRLB.

P. Stoica and A. Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," in *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 37, no. 5, pp. 720-741, May 1989.

# B. Capon´s method

- Capon´s method consists of the use of the MVDR beamforming solution to obtain the angles of arrival.

- Consider the signal model of a ULA with N sensors expressed by

$$\boldsymbol{x}[i] = \sum_{d=1}^{D} s_d[i]\mathbf{a}(\theta_d) + \boldsymbol{n}[i] \,, \in \mathbb{C}^N,$$

- Consider the design of the MVDR beamformer given by

$$\boldsymbol{w_o} = \arg\ \min\ \boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}\,, \qquad \text{subject to}\ \boldsymbol{w}^H\boldsymbol{a}(\theta_d) = 1$$

- The expression of the MVDR is described by

$$\boldsymbol{w_o} = \frac{\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)}{\boldsymbol{a}^H(\theta_d)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)}$$

# Direction finding with Capon´s method

- In order to estimate the direction of arrival of the signals we substitute the MVDR solution in the cost function, which yields:

$$MV = E[\,|\boldsymbol{w_o}^H \boldsymbol{x}[i]|^2] = \boldsymbol{w_o}^H \boldsymbol{R} \boldsymbol{w_o}$$
$$= \frac{1}{\boldsymbol{a}^H(\theta_d)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)}$$

- Capon´s spatial spectrum is then defined as:

$$P(\theta_d) = \frac{1}{\boldsymbol{a}^H(\theta_d)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)}$$

- The direction of arrival of all signals can be obtained by

$$\hat{\theta}_d = \arg\max_{\theta_d} P(\theta_d),\ \ d = 1,2,\ \ldots,\ \ D,$$

which performs a search over all angles in a grid according to a chosen spacing in degrees.

# Summary of the Capon algorithm

Initialization:

$N$ – number of sensors

$D$ – number of source signals to find

array geometry: ULA, UCA, UPA, etc

Computations:

for i = 1, 2, … Do

estimate $\boldsymbol{R}$: $\widehat{\boldsymbol{R}}\,[i] = \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}[l]\boldsymbol{x}^{H}\,[l]$

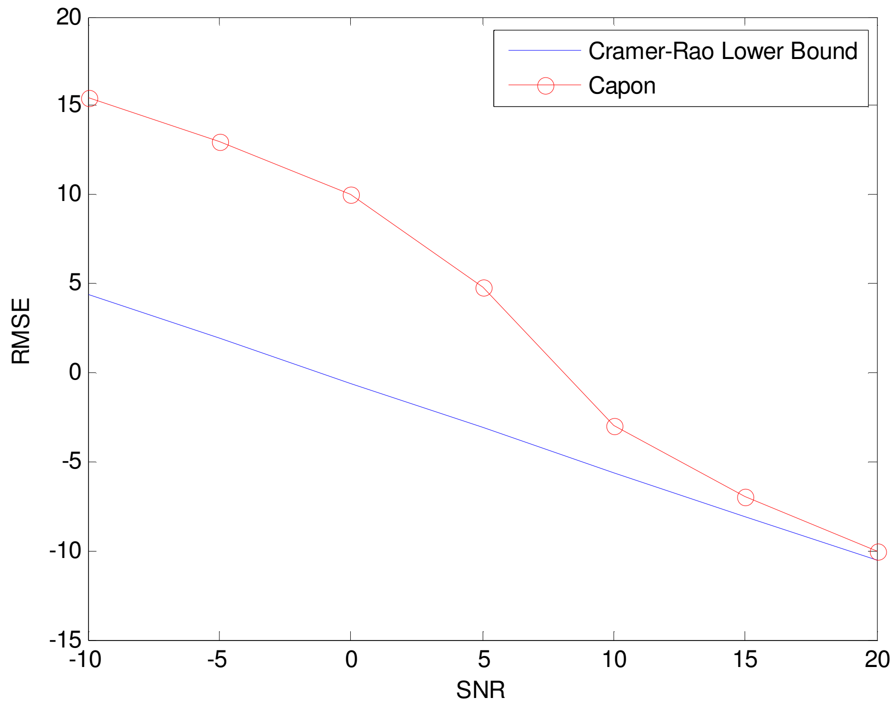
compute the Capon spectrum: $P(\theta_d) = \frac{1}{\boldsymbol{a}^{H}(\theta_d)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)}$

estimate the angles $\theta_d$: $\hat{\theta}_d = \arg\max_{\theta_d} P(\theta_d),\ \ d\ =\ 1,2,\dots,D$

end

# Example: direction finding with Capon´s method

- Scenario:
  - N = 10 sensors
  - I = 20 snapshots
  - D = 2 source signals with 35 and 40 degrees



# Root-Capon´s method

- One of the disadvantages of Capon´s method is the need for a search in a grid, which can be costly.

- An approach to avoid the search is based on a root technique, which requires the sensor array to be a ULA.

- Consider the spatial spectrum given by

$$P(\theta_d) = \frac{1}{\boldsymbol{a}^H(\theta_d)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)} = \frac{1}{C(\theta_d)},$$

where $C(\theta_d) = \boldsymbol{a}^H(\theta_d)\boldsymbol{R}^{-1}\boldsymbol{a}(\theta_d)$ is the null spectrum.

- We should take into account that finding the peaks of $P(\theta_d)$ is equivalent to finding the nulls of $C(\theta_d)$.

# Polynomial rooting

- The main idea behind root techniques is to write $\mathrm{C}(\theta_d)$ as a polynomial and then find the roots of the polynomial.

- To this end, let us start by writing the steering vector as

$$\boldsymbol{a}(\theta_d) = [1\ e^{-j2\pi\frac{d}{\lambda}sen\theta_d} \ldots e^{-j2\pi\frac{d}{\lambda}(N-1)sen\theta_d}]^T$$
$$= [1\ e^{-jf(\theta_d)} \ldots e^{-j(N-1)f(\theta_d)}]^T$$

- By using $z = e^{-jf(\theta_d)}$, we can write $\mathrm{C}(\theta_d)$ as

$$\mathrm{C}(z) = \sum_{j=0}^{N-1}\sum_{n=0}^{N-1} \mathrm{z}^{\mathrm{j}}\, \mathrm{r}_{\mathrm{jn}} \mathrm{z}^{-\mathrm{n}}$$
$$= \sum_{j=0}^{N-1}\sum_{n=0}^{N-1} \mathrm{r}_{\mathrm{jn}} \mathrm{z}^{\mathrm{j-n}},$$

where $\mathrm{r}_{\mathrm{jn}}$ is the jth element of $\boldsymbol{R}^{-1}$

# Polynomial rooting

- By defining $k = j - n$ and rewriting $C(z)$ as

$$\mathrm{C}(z) = \sum_{k=-N+1}^{N-1} q_k z^k,$$

where

$$q_k = \begin{cases} \sum_{n=0}^{k+N-1} r_{k-n,n}, & k = -(N-1), \dots, -1,0 \\ \sum_{n=k}^{N-1} r_{k-n,n}, & k = 1,2, \dots, (N-1) \end{cases}$$

# Hermitian property of $\boldsymbol{R}^{-1}$:

- The coefficients of $\mathrm{C}(z)$ are conjugate symmetric, i.e., $q_{-k} = {q_k}^*$

- We can then write the null spectrum as

$$\mathrm{C}(z) = \sum_{k=-N+1}^{N-1} {q_k}^* z^k,$$

and $\mathrm{C}(\theta_d) = \mathrm{C}(z)|_{z=e^{-jf(\theta_d)}}$

- The spectrum nulls of $\mathrm{C}(\theta_d)$ are due to the roots of $\mathrm{C}(z)$, which are close to the unit circle.

# Direction finding with root-Capon´s method

- In order to estimate the directions of arrival of the signals, we proceed as follows:
  - We compute the null spectrum of $C(z)$
  - We calculate the D closest roots of $C(z)$ to the unit circle
  - We compute $\theta_d = \sin^{-1}\left[\left(\frac{\lambda}{2\pi d}\right) \arg z_d\right]$, $d = 1,2,\dots,D$,

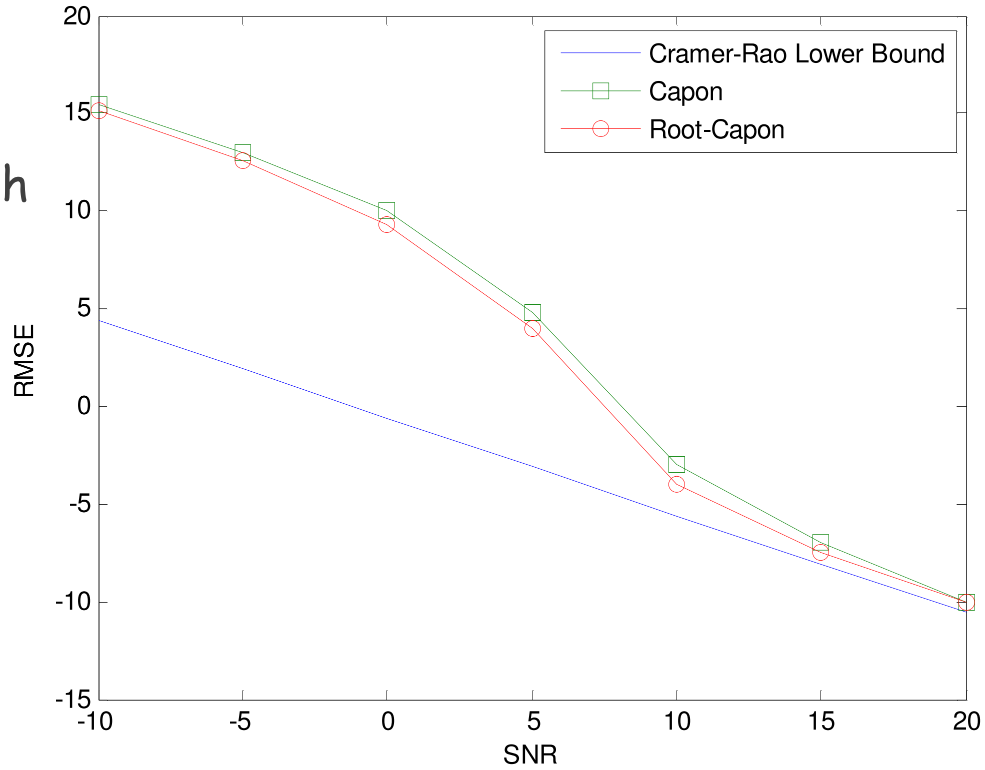
where $\arg z_d$ computes the angle associated with the complex root $z_d$.

# Important notes about Capon and root Capon´s methods

- Capon and root-Capon are not efficient estimators -> they do not approach the CRLB.
- Capon and root-Capon are effective in scenarios where the DoAs are sufficiently separated.
- When the DoAs are very close, one has to resort to ML or subspace methods.

# Example: Root-Capon performance

- Scenario:
  - N = 10 sensors
  - I = 20 snapshots
  - D = 2 source signals with 35 and 40 degrees



# C. MUSIC

- Key ideas:
  - exploitation of the structure of the data model $\boldsymbol{x}[i]$ and
  - the eigenstructure of the covariance matrix $\mathbf{R} = \mathrm{E}[\boldsymbol{x}[i]\boldsymbol{x}^H[i]]$.

- Requirements:
  - number of source signals D
  - covariance matrix $\mathbf{R} = \mathrm{E}[\boldsymbol{x}[i]\boldsymbol{x}^H[i]]$

Schmidt, R.O, "Multiple Emitter Location and Signal Parameter Estimation," IEEE Trans. Antennas Propagation, Vol. AP-34 (March 1986), pp.276-280. (original conf. version is from 1979!)

# Signal model and statistics

- Consider the signal model of a ULA with N sensors expressed by

$$\boldsymbol{x}[i] = \sum_{d=1}^{D} s_d[i]\mathbf{a}(\theta_d) + \boldsymbol{n}[i] \,, \in \mathbb{C}^N,$$

- The covariance matrix of the sensor array data is given by

$$\mathbf{R} = \mathrm{E}\big[\boldsymbol{x}[i]\boldsymbol{x}^H[i]\big] = \mathbf{A}\underbrace{\mathrm{E}\big[\boldsymbol{s}[i]\boldsymbol{s}^H[i]\big]}_{\boldsymbol{R}_s}\boldsymbol{A}^H + \underbrace{\mathrm{E}\big[\boldsymbol{n}[i]\boldsymbol{n}^H[i]\big]}_{\boldsymbol{R}_n = \boldsymbol{\sigma}_n{}^2\boldsymbol{I}}$$

$$= \mathbf{A}\boldsymbol{R}_s\boldsymbol{A}^H + \boldsymbol{\sigma}_n{}^2\boldsymbol{I}$$

- The covariance matrix can be estimated via time averages:

$$\hat{\mathbf{R}}[i] = \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}[l]$$

# Eigendecomposition and model-order selection

- Consider an eigendecomposition of the covariance matrix of the sensor array data:

$$\mathbf{R} = \mathrm{E}\big[\boldsymbol{x}[i]\boldsymbol{x}^H[i]\big] = \boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H + \boldsymbol{\sigma_n}^2\boldsymbol{I} = \sum_{n=1}^{N} \lambda_n \boldsymbol{\phi}_n \boldsymbol{\phi}_n{}^H$$

$$= [\boldsymbol{\phi}_1|\boldsymbol{\phi}_2| \quad \dots \quad |\boldsymbol{\phi}_N] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N \end{bmatrix} [\boldsymbol{\phi}_1|\boldsymbol{\phi}_2| \quad \dots \quad |\boldsymbol{\phi}_N]^H$$

where R is assumed to be full rank.

- The matrix $\boldsymbol{R}_s$ is nonsingular and guarantees that $\boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H$ is positive semi-definite with rank D.
- N-D eigenvalues of $\boldsymbol{A}\boldsymbol{R}_s\boldsymbol{A}^H$ are nonzero and equal to $\boldsymbol{\sigma_n}^2$.

# Eigendecomposition and model-order selection

- Consider the covariance matrix of the sensor array data expressed as:

$$\mathbf{R} = \sum_{n=1}^{D} \lambda_n \boldsymbol{\phi}_n \boldsymbol{\phi}_n{}^H + \sum_{k=D+1}^{N} \lambda_k \boldsymbol{\phi}_k \boldsymbol{\phi}_{\boldsymbol{k}}{}^H$$
$$= \sum_{n=1}^{D} \lambda_n \boldsymbol{\phi}_n \boldsymbol{\phi}_n{}^H + \sigma_n{}^{\mathbf{2}} \boldsymbol{I},$$

where $\lambda_k = \sigma_n{}^{\mathbf{2}}$ for $k = D + 1, \ldots, N$.

- The number of sources D can be estimated by

$$\widehat{D} = \mathrm{N} - \mathrm{K},$$

Where $K$ is the multiplicity of $\lambda_k = \sigma_n{}^{\mathbf{2}}$.

# Signal and noise subspaces

- Let us write the covariance matrix as a function of the signal and noise subspaces as given by

$$\mathbf{R} = \sum_{n=1}^{N} \lambda_n \boldsymbol{\phi}_n \boldsymbol{\phi}_n{}^H = \sum_{n=1}^{D} \lambda_n \boldsymbol{\phi}_n \boldsymbol{\phi}_n{}^H + \sum_{k=D+1}^{N} \lambda_k \boldsymbol{\phi}_k \boldsymbol{\phi}_{\boldsymbol{k}}{}^H$$

$$= [\boldsymbol{\phi}_1 | \boldsymbol{\phi}_2 | \quad \dots \quad | \boldsymbol{\phi}_D] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_D \end{bmatrix} [\boldsymbol{\phi}_1 | \boldsymbol{\phi}_2 | \quad \dots \quad | \boldsymbol{\phi}_D]^H$$

$$+ [\boldsymbol{\phi}_{D+1} | \boldsymbol{\phi}_{D+2} | \quad \dots \quad | \boldsymbol{\phi}_N] \begin{bmatrix} \lambda_{D+1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N \end{bmatrix} [\boldsymbol{\phi}_{D+1} | \boldsymbol{\phi}_{D+2} | \quad \dots \quad | \boldsymbol{\phi}_N]^H$$

$$= \boldsymbol{\phi}_S \boldsymbol{\Lambda}_S \boldsymbol{\phi}_S{}^H + \boldsymbol{\phi}_N \boldsymbol{\Lambda}_N \boldsymbol{\phi}_N{}^H,$$

where in the signal subspace we have $\boldsymbol{\phi}_S \in \mathbb{C}^{N \times D}$ and $\boldsymbol{\Lambda}_S \in \mathbb{C}^{D \times D}$ and in the noise subspace we have $\boldsymbol{\phi}_{N \in \mathbb{C}^{N \times N-D}}$ and $\boldsymbol{\Lambda}_N \in \mathbb{C}^{N-D \times N-D}$.

# Signal and noise subspaces

- The eigenvector associated with the eigenvalue $\lambda_k$ is the vector $\boldsymbol{\phi}_k$ such that

$$(\boldsymbol{R} - \lambda_k \boldsymbol{I})\boldsymbol{\phi}_k = \boldsymbol{0} \ , k = D+1, \dots, N$$

- For the eigenvectors associated with the N-D smallest eigenvalues, we have

$$(\boldsymbol{R} - \lambda_k \boldsymbol{I})\boldsymbol{\phi}_k = (\mathbf{A}\boldsymbol{R}_s\boldsymbol{A}^H + \boldsymbol{\sigma_n}^2\boldsymbol{I} - \underbrace{\lambda_k}_{\sigma_n^2} \boldsymbol{I})\boldsymbol{\phi}_k$$

$$= \mathbf{A}\boldsymbol{R}_s\boldsymbol{A}^H\boldsymbol{\phi}_k = \boldsymbol{0}$$

- Since $\mathbf{A}$ has full column rank and $\boldsymbol{R}_s$ is non singular, this implies that

$$\boldsymbol{A}^H\boldsymbol{\phi}_k = \boldsymbol{0}$$

- This means that the eigenvectors associated with the N-D smallest eigenvalues are orthogonal to the D steering vectors $\mathrm{a}(\theta_d)$.

# Subspace processing principle

- From the condition $\boldsymbol{A}^H \boldsymbol{\phi}_k = \mathbf{0}$, we have

$$\mathbf{a}^H(\theta_d)\boldsymbol{\phi}_S{\boldsymbol{\phi}_S}^H\mathbf{a}(\theta_d) = \|\mathbf{a}^H(\theta_d)\boldsymbol{\phi}_S\|^2$$
$$= \sum_{d=1}^{D} |\mathbf{a}^H(\theta_d)\boldsymbol{\phi}_d|^2 \neq 0,$$

and

$$\|\mathbf{a}^H(\theta_i)\boldsymbol{\phi}_N\|^2 = 0, i = 1,2, \dots, D$$

- The theory of subspace processing follows from the above relation and amounts to estimating the steering vectors by exploiting their orthogonality to the noise subspace.

# MUSIC algorithm

- In order to employ the MUSIC algorithm, we employ the noise subspace

$$\boldsymbol{\phi}_N = [\boldsymbol{\phi}_{D+1} | \boldsymbol{\phi}_{D+2} | \quad \dots \quad | \boldsymbol{\phi}_N] \in \mathbb{C}^{N \times N-D}$$

and we form the function given by

$$C(\theta_d) = \mathbf{a}^H(\theta_d)\, \boldsymbol{\phi}_N \boldsymbol{\phi}_N{}^H\, \mathbf{a}(\theta_d),$$

where the orthogonality between the signal and the noise subspaces will be exploited to estimate $\theta_d$.

- The relation $\mathbf{a}^H(\theta_d)\boldsymbol{\phi}_N \boldsymbol{\phi}_N{}^H \mathbf{a}(\theta_d) = 0$ will hold when $\theta_d$ corresponds to the actual angle of arrival.

# MUSIC algorithm

- We then construct the MUSIC spatial spectrum:

$$P_{MU}(\theta_d) = \frac{1}{\boldsymbol{a}^H(\theta_d)\,\boldsymbol{\phi}_N \boldsymbol{\phi}_N{}^H \boldsymbol{a}(\theta_d)} = \frac{1}{C_{MU}(\theta_d)}$$

$$= \frac{1}{\boldsymbol{a}^H(\theta_d)\left(\boldsymbol{I} - \boldsymbol{\phi}_S \boldsymbol{\phi}_S{}^H\right) \boldsymbol{a}(\theta_d)}$$

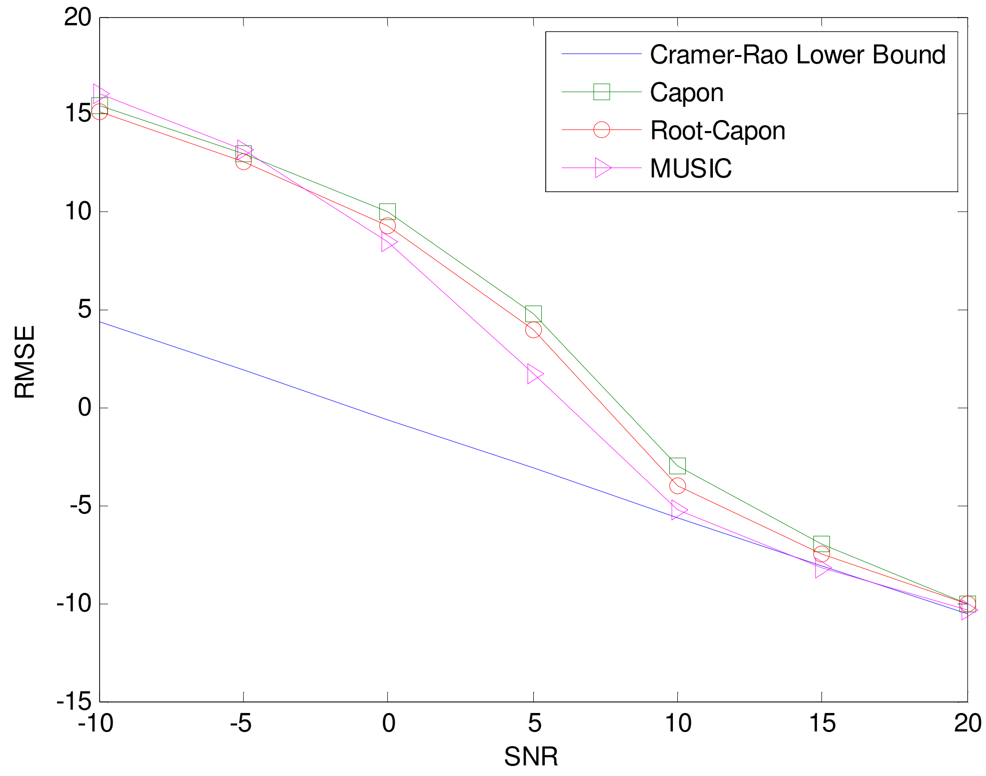
- The direction of arrival estimation is obtained by

$$\hat{\theta}_d = \arg\max_{\theta_d} P_{MU}(\theta_d), \;\; \mathrm{d} = 1, 2, \;\ldots, \;\; D,$$

where the cost depends on the angle spacing.

# Example: MUSIC algorithm

- Scenario:
  - N = 10 sensors
  - I = 20 snapshots
  - D is assumed known
  - D = 2 source signals with 35 and 40 degrees



# Summary of the MUSIC algorithm

Initialization:

$N$ – number of sensors

$D$ – number of source signals (optional)

array geometry: ULA, UCA, UPA, etc

Computations:

for i = 1, 2, ... Do

estimate $\boldsymbol{R}$: $\hat{\boldsymbol{R}}\,[i] = \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}[l]\boldsymbol{x}^H\,[l]$

perform EVD: $\hat{\boldsymbol{R}}\,[i] = \sum_{n=1}^{N} \lambda_n \boldsymbol{\phi}_n\, \boldsymbol{\phi}_n{}^H$

estimate D: $\hat{D} = N - K$

compute the MUSIC spectrum: $P_{MU}(\theta_d) = \frac{1}{\boldsymbol{a}^H(\theta_d)\, \boldsymbol{\phi}_N \boldsymbol{\phi}_N{}^H \boldsymbol{a}(\theta_d)}$

estimate the angles $\theta_d$: $\hat{\theta}_d = \arg\max_{\theta_d} P_{MU}(\theta_d),\ d = 1,2,\ldots,D$

end

# Root-MUSIC

- The main idea behind root techniques is to write $C(\theta_d)$ as a polynomial and then find the roots of the polynomial.

- To this end, let us start by writing the steering vector as

$$\boldsymbol{a}(\theta_d) = [1\ e^{-j2\pi\frac{d}{\lambda}sen\theta_d} \dots e^{-j2\pi\frac{d}{\lambda}(N-1)sen\theta_d}]^T$$
$$= [1\ e^{-jf(\theta_d)} \dots e^{-j(N-1)f(\theta_d)}]^T$$

- By using $z = e^{-jf(\theta_d)}$ and $a_{\boldsymbol{n}}(\theta_d) = e^{-j2\pi n\frac{d}{\lambda}sen\theta_d}$, we can write $C_{MU}(\theta_d)$ as

$$C_{MU}(\theta_d) = \boldsymbol{a}^H(\theta_d)\ \boldsymbol{\phi}_N\boldsymbol{\phi}_N{}^H\boldsymbol{a}(\theta_d)$$
$$= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} e^{-j2\pi n\frac{d}{\lambda}sen\theta_d}\ \boldsymbol{\phi}_{lk}e^{j2\pi n\frac{d}{\lambda}sen\theta_d}$$
$$= \sum_{l=0}^{N-1}\sum_{k=0}^{N-1} \boldsymbol{\phi}_{lk}\mathrm{z}^{\mathrm{l}-k},$$

where$\boldsymbol{\phi}_{lk}$is the lkth element of $\boldsymbol{\phi}_N\boldsymbol{\phi}_N{}^H$.

# Root-MUSIC

- By defining $\mathrm{m} = l - k$ and rewriting $\mathrm{C}_{MU}(\theta_d)$, we have

$$\mathrm{C}_{MU}(z) = \sum_{m=-N+1}^{N-1} \boldsymbol{\phi}_m z^m = \sum_{m=-N+1}^{N-1} \boldsymbol{\phi}_m e^{j2\pi n\frac{d}{\lambda}sen\theta_d},,$$

where $\boldsymbol{\phi}_m = \sum_{m=-N+1}^{N-1} \boldsymbol{\phi}_{mn}$ is the sum of entries of $\boldsymbol{\phi}_N \boldsymbol{\phi}_N{}^H$.

- The direction of arrival estimation can be obtained by

$$\theta_d = \sin^{-1}\left[\left(\frac{\lambda}{2\pi d}\right)\arg z_d\right], d = 1,2, \dots, D,$$

where $\arg z_d$ computes the angle associated with the complex root $z_d$.

# Important notes on Root-MUSIC

- The performance of root-MUSIC is superior to MUSIC due to the polynomial rooting, which is more precise than a search.
- MUSIC and root-MUSIC are efficient estimators -> they converge to the CRLB.

# Summary of the root-MUSIC algorithm

Initialization:

$N$ – number of sensors
$D$ – number of source signals (optional)
array geometry: ULA

Computations:

for i = 1, 2, ... Do

estimate $\boldsymbol{R}$: $\widehat{\boldsymbol{R}}\,[i] = \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}[l]\boldsymbol{x}^H\,[l]$

perform EVD: $\widehat{\boldsymbol{R}}\,[i] = \sum_{n=1}^{N} \lambda_n \boldsymbol{\phi}_n\, \boldsymbol{\phi}_n{}^H$

estimate D: $\widehat{D} = N - K$

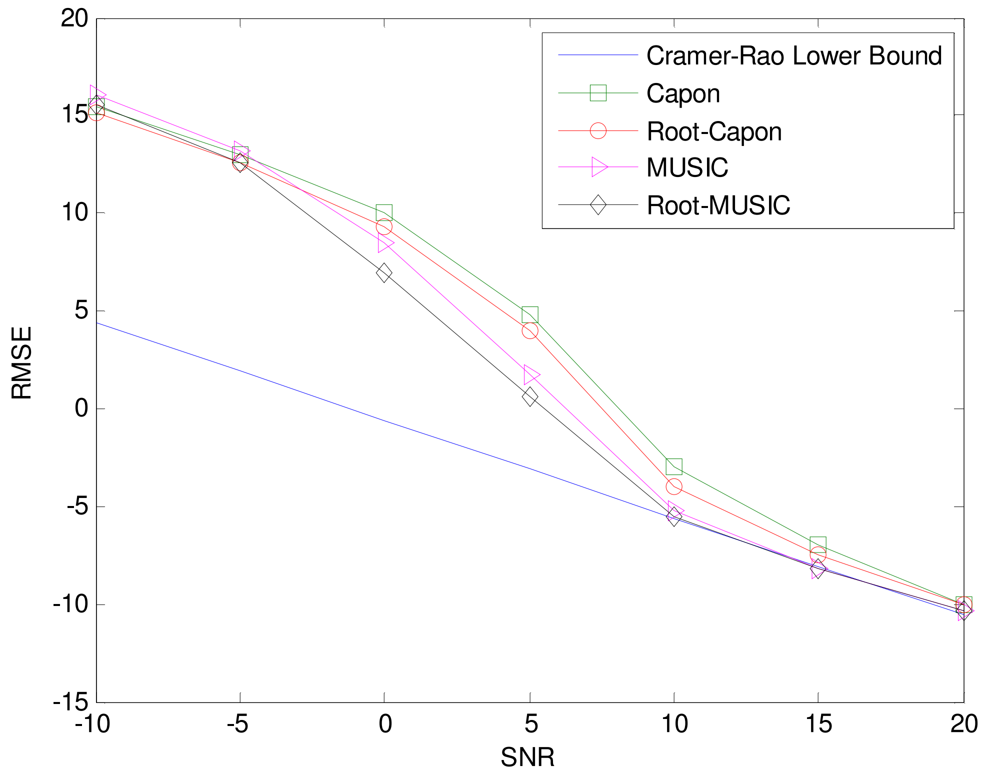
compute and factorize the $C_{MU}(z) = \boldsymbol{a}^H(\theta_d)\, \boldsymbol{\phi}_{\boldsymbol{N}}\boldsymbol{\phi}_{\boldsymbol{N}}{}^H \boldsymbol{a}(\theta_d)$

estimate the angles $\theta_d$:$\theta_d = \sin^{-1}\left[\left(\frac{\lambda}{2\pi d}\right)\arg z_d\right]$, $d = 1,2,\dots,D$,

end

# Example: Root-MUSIC algorithm

- Scenario:
  - N = 10 sensors
  - I = 20 snapshots
  - D is assumed known
  - D = 2 source signals with 35 and 40 degrees



# D. ESPRIT

- Main idea:
  - to split a sensor array that must be centro-symmetric into 2 identical subarrays.
  - exploit the shift invariance property of the array.

- Consider the signal model of a ULA with N sensors expressed by

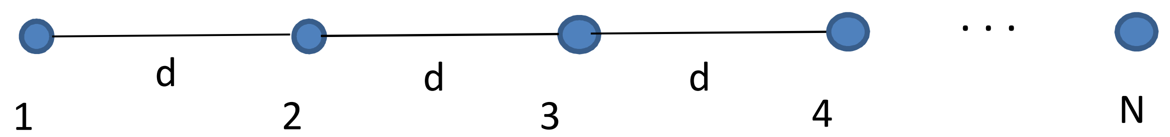
$$\boldsymbol{x}[i] = \sum_{d=1}^{D} s_d[i]\mathbf{a}(\theta_d) + \boldsymbol{n}[i] , \in \mathbb{C}^N$$
$$= \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}[i] + \boldsymbol{n}[i],$$

where $\boldsymbol{a}(\theta_d) = [1 \; e^{-j2\pi\frac{d}{\lambda}sen\theta_d} \dots e^{-j2\pi\frac{d}{\lambda}(N-1)sen\theta_d}]^T$

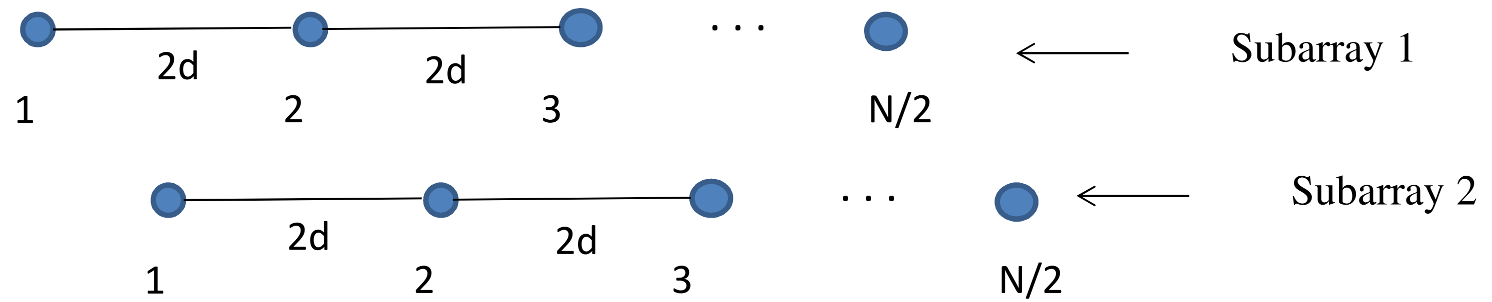
R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," in IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 37, no. 7, pp. 984-995, Jul 1989.

# Subarray splitting

- Consider the original centro-symmetric sensor array:




- We can split it into two identical subarrays by, for example, selecting every other sensor for building the subarrays.



# Assumptions

- The sensor array is in the far field
- The signals are wide-sense stationary
- Centro-symmetric array -> shift-invariance structure

# Relations between subarrays

- Let us express a relation between the steering vectors of the 2 subarrays:

$$\mathbf{a_2}(\theta_d) = \mathbf{a_1}(\theta_d) e^{-j2\pi \frac{d}{\lambda} sin\theta_d}$$

- If we consider the array manifold of the subarrays then we have

$$\mathbf{A_2}(\theta) = \mathbf{A_1}(\theta)\boldsymbol{\Psi} \in \mathbb{C}^{\frac{\boldsymbol{N}}{\mathbf{2}} \times D},$$

where $\boldsymbol{\Psi} = \begin{bmatrix} e^{-j2\pi \frac{d}{\lambda} sen\theta_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & e^{-j2\pi \frac{d}{\lambda} sen\theta_D} \end{bmatrix} \in \mathbb{C}^{D \times D}$

# Further relations

- Consider the array manifold matrix $\mathbf{A}(\theta)$ of the original sensor array, we can also write

$$\mathbf{A_1}(\theta) = \boldsymbol{J_1}\mathbf{A}(\theta) \text{ and } \mathbf{A_2}(\theta) = \boldsymbol{J_2}\mathbf{A}(\theta),$$

where the selection matrices are given by

$$\boldsymbol{J_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{C}^{\boldsymbol{N/2\times N}}$$

and

$$\boldsymbol{J_2} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{C}^{\boldsymbol{N/2\times N}}$$

# Further relations

- We can then write the following relation between the subarrays:

$$\boldsymbol{J}_2\mathbf{A}(\theta) = \mathbf{A}_2(\theta) = \mathbf{A}_1(\theta)\boldsymbol{\Psi}$$
$$= \boldsymbol{J}_1\mathbf{A}(\theta)\boldsymbol{\Psi}$$

← Invariance equation

- Problem: $\mathbf{A}(\theta)$ is unknown and contains the angles $\theta_d$. How can we estimate these angles?

- Solution: subspace processing techniques

# ESPRIT algorithm

- Consider again the signal model given by

$$\boldsymbol{x}[i] = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{s}[i] + \boldsymbol{n}[i],$$

where

$$\mathbf{R} = \mathrm{E}\big[\boldsymbol{x}[i]\boldsymbol{x}^H[i]\big] = \mathbf{A}(\boldsymbol{\theta})\boldsymbol{R_s}\boldsymbol{A^H}(\boldsymbol{\theta}) + \sigma_{\boldsymbol{n}}{}^2\boldsymbol{I} = \boldsymbol{\phi}\big(\boldsymbol{\Lambda} + \boldsymbol{\sigma_n}{}^2\boldsymbol{I}\big)\boldsymbol{\phi^H} =$$

$$[\boldsymbol{\phi}_S \quad \boldsymbol{\phi}_N]\begin{bmatrix} \lambda_1 + \sigma_{\boldsymbol{n}}{}^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \lambda_D + \sigma_{\boldsymbol{n}}{}^2 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma_{\boldsymbol{n}}{}^2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & \sigma_{\boldsymbol{n}}{}^2 \end{bmatrix}[\boldsymbol{\phi}_S \quad \boldsymbol{\phi}_N]^H.$$

- Suppose now that we introduce a D x D invertible matrix $\boldsymbol{T}$ in the relation:

$$\boldsymbol{A}(\boldsymbol{\theta}) = \boldsymbol{\phi}_S\boldsymbol{T} \Rightarrow \boldsymbol{\phi}_S = \boldsymbol{A}(\boldsymbol{\theta})\boldsymbol{T}^{-1}$$

# ESPRIT algorithm

- The invariance equation is given by

$$J_2\mathbf{A}(\theta) = J_1\mathbf{A}(\theta)\boldsymbol{\Psi},$$

and we can also write

$$J_2\boldsymbol{\phi}_S = J_1\boldsymbol{\phi}_S\boldsymbol{T}\boldsymbol{\Psi}\boldsymbol{T}^{-1}$$

- A least-squares solution to the invariance equations considers $D^2$ unknowns and $N/2$ equations is described by

$$\boldsymbol{T}\boldsymbol{\Psi}\boldsymbol{T}^{-1} = \left(\boldsymbol{\phi}_S{}^{H} J_1{}^{H} J_1\boldsymbol{\phi}_S\right)^{-1}\boldsymbol{\phi}_S{}^{H} J_1{}^{H} J_2\boldsymbol{\phi}_S$$

# ESPRIT algorithm

- By performing an eigendecomposition on $\boldsymbol{T\Psi T}^{-1}$ we obtain the D eigenvalues corresponding to the angles of arrival:

$$\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_D$$

- The direction of arrival of the signals can then be computed by

$$\theta_d = \left(\frac{\lambda}{2\pi d}\right) \sin^{-1}[\arg \hat{\lambda}_d]$$

# Summary of the ESPRIT algorithm

Initialization:

$N$ – number of sensors
$D$ – number of source signals
array geometry: ULA, subarrays, $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$

Computations:

for i = 1, 2, ... Do

estimate $\boldsymbol{R}$: $\widehat{\boldsymbol{R}}\,[i] = \frac{1}{i}\sum_{l=1}^{i} \boldsymbol{x}[l]\boldsymbol{x}^H\,[l]$

perform EVD: $\widehat{\boldsymbol{R}}\,[i] = \boldsymbol{\phi}\boldsymbol{\Lambda}\boldsymbol{\phi}^{\boldsymbol{H}}$, where $\boldsymbol{\phi} = [\boldsymbol{\phi}_S \quad \boldsymbol{\phi}_N]$

compute the invariance equation:

$$\boldsymbol{T\Psi T}^{-1} = \left(\boldsymbol{\phi}_S{}^{\boldsymbol{H}}\boldsymbol{J}_1{}^{\boldsymbol{H}}\boldsymbol{J}_1\boldsymbol{\phi}_S\right)^{-1}\boldsymbol{\phi}_S{}^{\boldsymbol{H}}\boldsymbol{J}_1{}^{\boldsymbol{H}}\boldsymbol{J}_2\boldsymbol{\phi}_S$$

compute the eigenvalues of $\boldsymbol{T\Psi T}^{-1}$: $\hat{\lambda}_1, \hat{\lambda}_2, \dots, \hat{\lambda}_D$

estimate the angles $\theta_d$:$\theta_d = \left(\frac{\lambda}{2\pi d}\right)\sin^{-1}[\,\arg \hat{\lambda}_d]$, $d = 1,2,\dots,D$,

end

# Example: ESPRIT algorithm

- Scenario:
  - N = 10 sensors
  - I = 20 snapshots
  - D is assumed known
  - D = 2 source signals with 35 and 40 degrees